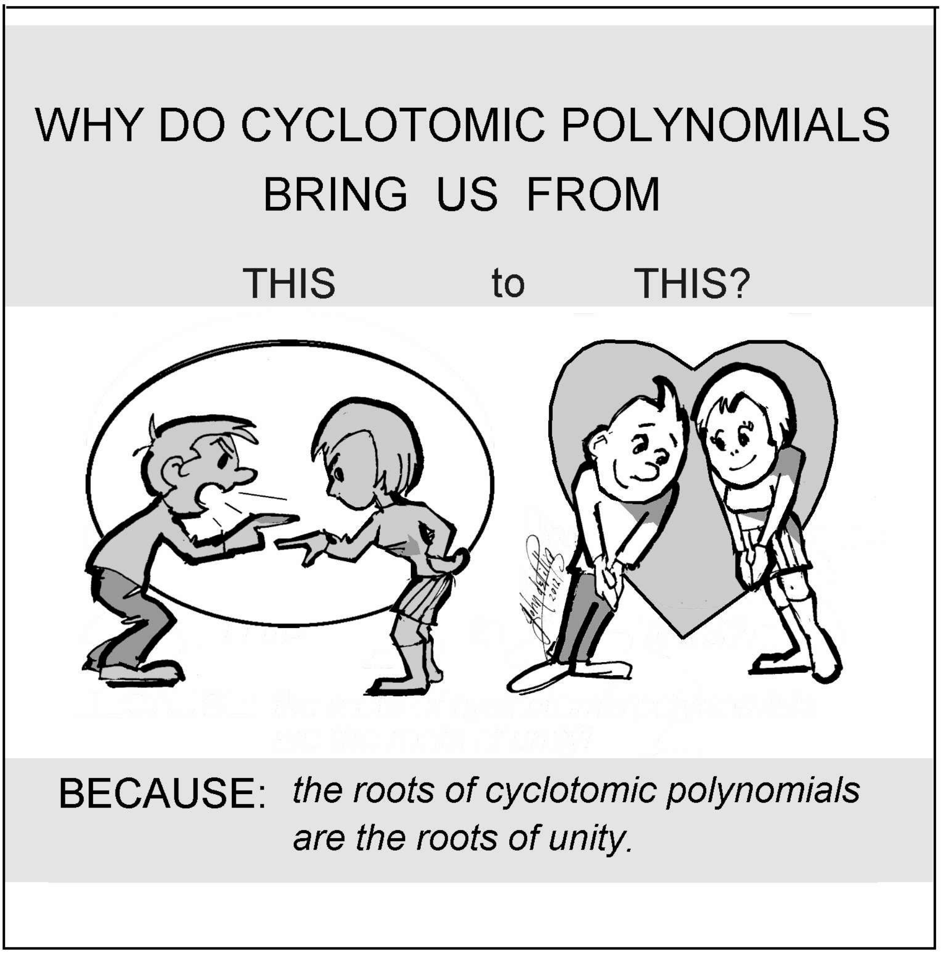
Vol. 85, No. 1, February 2012

MAA

MATHEMATICS MAGAZINE



Mathematics brings us together

- Similarity vs. Complementarity in Chinese Mathematics
- Eisenstein Triples
- Pizza, a Social Network, First Digits, and more

EDITORIAL POLICY

Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

A more detailed statement of author guidelines appears in this *Magazine*, Vol. 83, at pages 73–74, and is available at the *Magazine*'s website www.maa.org/pubs/mathmag.html. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, or published by another journal or publisher.

Please submit new manuscripts by email directly to the editor at mathmag@maa.org. A brief message containing contact information and with an attached PDF file is preferred. Word-processor and DVI files can also be considered. Alternatively, manuscripts may be mailed to Mathematics Magazine, 132 Bodine Rd., Berwyn, PA 19312-1027. If possible, please include an email address for further correspondence.

Cover image by John dePillis.

MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Lancaster, PA, bimonthly except July/August.
The annual subscription price for *MATHEMATICS MAGAZINE* to an individual member of the Association is $131. Student and unemployed members receive a 66% dues discount; emeritus members receive a 50% discount; and new members receive a 20% dues discount for the first two years of membership.)

Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036. Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

Advertising correspondence should be addressed to

MAA Advertising
1529 Eighteenth St. NW
Washington DC 20036

Phone: (877) 622-2373
E-mail: tmarmor@maa.org

Further advertising information can be found online at www.maa.org

Change of address, missing issue inquiries, and other subscription correspondence:

MAA Service Center, *maahq@maa.org*

All at the address:

The Mathematical Association of America
1529 Eighteenth Street, N.W.
Washington, DC 20036

Periodicals postage paid at Washington, D.C. and additional mailing offices.

Postmaster: Send address changes to Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036-1385.

Printed in the United States of America

Vol. 85, No. 1, February 2012

MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

We thank John de Pillis for the cover image, which suggests the theme that "mathematics brings us together." Actually that could be a theme of every issue.

Frank Swetz's article brings us together with mathematicians working over a millennium ago. In his book, *Was Pythagoras Chinese*, he showed us the achievements of ancient Chinese mathematicians involving right triangles. Many of their arguments seem to rely on similar triangles—even though the Chinese were not using similarity in other ways. How could this be? In this article he reveals their secret: the arguments were based on area comparisons. In one example at least, the area comparison yields results as efficiently as any synthetic technique a modern researcher is likely to find.

If you like right triangles, perhaps you should try triangles with 60-degree angles. When they have integer sides, Russell Gordon calls them "Eisenstein triples," and he describes their properties starting at page 12. Want to find an Eisenstein triple involving your lucky number? Start by factoring.

In the Notes, Ken Ross tells us about the distributions of the first digits of certain sequences of numbers. They're just Benford Distributions, right? Go look. You'll also find an article by Greg Frederickson about dissections of a pizza, and some new ways of looking at L'Hospital's rule, the totient function, and Galois connections. You never know when you'll need the LaTeX "verse" environment. One of our authors has used it for a theorem.

You'll also find the most recent Putnam problems and their solutions, available because of the lightning-fast work of the Putnam committee. Of course, you solved all the problems on the day they were announced, but now you can compare the committee's solutions to your own.

At the very end of this issue, in the Letters to the Editors, you'll find one more reason to nurture your back issues of the MAA publications and to read them frequently.

Walter Stromquist, Editor

ARTICLES

Similarity vs. the "In-and-Out Complementary Principle": A Cultural Faux Pas

FRANK J. SWETZ
The Pennsylvania State University Harrisburg
Middletown, PA 17057
fjs2@psu.edu

Whenever I undertake historical investigation involving other times and cultures, I attempt to employ the contemporary mathematical modes of understanding that existed in these situations. I know that mathematics has been done in many ways and I respect the intellectual efforts that have produced techniques and concepts different from those that are popular today and that I would choose to use myself.

One of my first cross-cultural investigations of historical mathematics involved an examination and analysis of the *Jiuzang suanshu* [Nine Chapters of the Mathematical Art] (ca. 100 BCE) [**8**], the preeminent Chinese mathematical classic. Due to limited experience and resources, I directed my efforts at translation and analysis of the ninth and final chapter of this work, the chapter that dealt with problems involving right triangles [*Gou gu*]. I felt that this chapter testified to the high level of right triangle understanding achieved by Chinese mathematicians at this early date and that its exposure to a larger audience would be enlightening. The result of this effort was the publishing of *Was Pythagoras Chinese?* in 1977 [**12**].

The book was well received. Among the encouraging reviews was one that I particularly appreciated; it was from a Chinese mathematical historian who noted that I had tried to do the problems in the manner of the ancient Chinese. Now, 40 years later, I realize that my mathematical interpretation of these problems and their solution techniques was incorrect.

In order to understand how this situation arose, one must appreciate the content and format of the early Chinese works. Like most early mathematical works, Chinese mathematical texts are usually collections of problems with their solution techniques given. They are very concise in their wording; little explanatory or instructional information is provided. Further, the authors assume that the readers have been sufficiently trained in the necessary computational techniques so that only very general solution outlines are needed. Bureaucratic users of the text mechanically applied the given formulae to obtain their desired results. However, on occasion, the contents and methods of such texts were examined in detail by more mathematically sophisticated scholars who tried to explain, justify, and correct where needed, the given procedures and results. The *Jiuzhang* was annotated by two such scholars: Liu Hui (3rd century CE) and Li Chunfeng (602–670 CE). In the later Yuan Dynasty, Yang Hui (1238–1298 CE) wrote a separate detailed commentary and analysis on the text [**17**].

Math. Mag. **85** (2012) 3–11. doi:10.4169/math.mag.85.1.3.

The misinterpretations

Consider problem 19 from the *Gou-gu* chapter, as it is given [**12**, p. 54] and as I interpreted it.

(Given) *A square walled city of unknown dimensions. There are gates at the center of each side. It is known that there is a tree 30 pu from the north gate, and standing 750 pu from the west gate one can see the tree. What are the dimensions of the city?*

(Answer) *300 pu or 1 li.*

(Method) *Obtain the product of 750 and 30. The dimensions of the city will be equal to the square root of the product of four, 750, and 30.*

[A *pu* or *bu* is a double pace of 6 *chi*, or Chinese feet; a *li* is a Chinese mile of 300 *bu*.]

In response to the given, I drew the diagram in FIGURE 1 and labeled it accordingly:

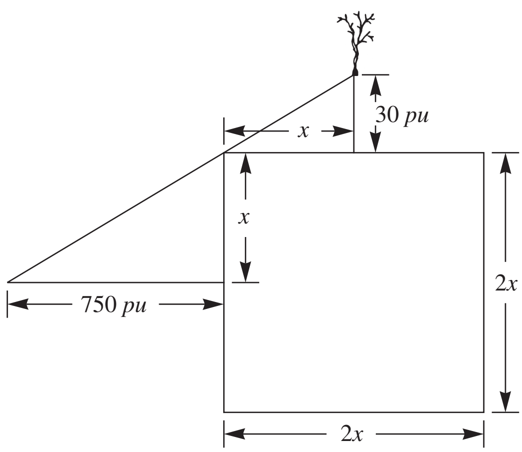


Figure 1 Problem 19 from the *Jiuzang suanshu*

From my high school mathematics training, I immediately noticed that the triangle ABC is similar to triangle CDE and developed a proportion involving their sides:

$$
\begin{aligned}
750/x &= x/30 \\
\Rightarrow \quad x^2 &= (30)(750) \\
\Rightarrow \quad 2x &= \sqrt{(4)(30)(750)}, \\
\Rightarrow \quad x &= 300\, pu \text{ or } 1\, li,
\end{aligned}
$$

which is the correct answer.

For me, and probably for any contemporary reader, the key to solving this problem was employing a similarity of triangles. (A similarity is defined by a transformation σ of the plane, such that for any two points P and Q, we have $d(\sigma(\mathrm{P}), \sigma(\mathrm{Q})) = k\, d(\mathrm{P}, \mathrm{Q})$ for some uniform positive constant k.) However, the Chinese did not use this concept. They used a technique called the "In-and-Out Complementary Principle" which visualized the situation geometrically as illustrated in FIGURE 2 below. Rather than looking for triangles that could be compared, they sought rectangles from which they could deduce mathematical relationships. In this situation the rectangle R_1 is equal in area to

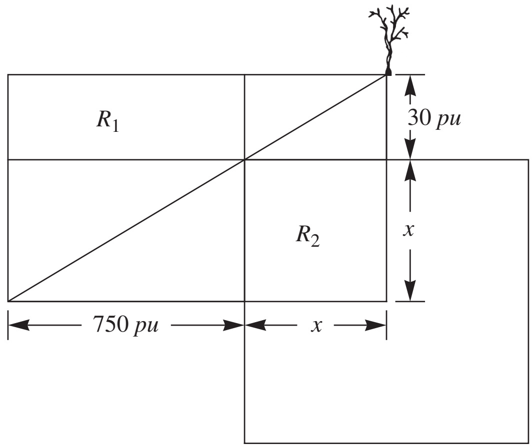


Figure 2 Problem 19 solved

rectangle R_2. Thus,

$$(30)(750) = x^2$$

and it follows as before that $2x = \sqrt{(4)(30)(750)}$ and $x = 300\,pu$.

An astute modern reader familiar with the history of mathematics, in studying the 24 problems of the *Gou-gu* chapter, will recognize a repeated use of "the rule of three," specifically, in problems: 17, 18, 19, 20, 22, 23, and 24. This is the principle that, from any three values in a proportional relationship such as $a/b = c/d$, one can determine the fourth. It has been a very powerful algebraic tool employed from ancient times well through the 18th century [**9**, vol. 2, pp. 283–290]. In each instance of the Chinese use of this solution technique in the problems listed above, the proportions established can be traced to rectangles found by an application of the In-and-Out Complementary Principle.

The In-and-Out Complementary Principle

Chinese mathematicians used an intuitive geometric algebra similar to that employed by the ancient Babylonians. They visualized the product of two numbers as the area of a rectangle. In a situation requiring algebraic computation, mathematicians would attempt to "package" unknowns and knowns into the form of rectangles and then seek out rectangles with the same area and work out proportional relationships. Right triangles were considered the regions obtained by diagonally bisecting a rectangle. Frequently mathematicians would conceptually or physically dissect plane regions, rearrange them and join the pieces into mathematically meaningful patterns, from which they could deduce necessary mathematical results. In their dissection activities, they realized that area was preserved under a set of planar transformations and that the disjoint union of area was additive. This is the essence of the IOCP.

Wu Wenjun was the first modern researcher to appreciate the significance of IOCP in traditional mathematics thinking. He published his findings in 1978 [**16**]. Wu traced back the existence of IOCP to a 3rd century commentary by the mathematician Zhao Shuang on the mathematical classic *Zhoubi* [**3**] and the "solar height problem" [**3**, pp. 218–221][**10**, pp. 39–41] and also identified its use in six mathematical classics [**15**, p. 66].

In a sense, this principle was articulated by Euclid as Proposition 43 in Book I of his Elements [**5**] and the Chinese conception of "similarity" is best explained by the use of a diagram (FIGURE 3). (Note that Euclid included parallelograms.) As shown, diagonal BD bisects □AC. (Rectangles will be designated by their left-to-right ascending diagonals.) The triangles on opposite side of the diagonal are congruent: $\triangle \mathrm{DJF} \cong \triangle \mathrm{FHD}$; $\triangle \mathrm{EBF} \cong \triangle \mathrm{GFB}$. The areas of the remaining rectangles are also equal:

$$\text{area}(\square \mathrm{AF}) = \text{area}(\square \mathrm{FC}).$$

All of these results follow:

$$\mathrm{AE} \times \mathrm{EF} = \mathrm{EF} \times \mathrm{GC}, \qquad \mathrm{AB} \times \mathrm{BG} = \mathrm{EB} \times \mathrm{BC} \ldots$$

$$\mathrm{JF} : \mathrm{EB} = \mathrm{DJ} : \mathrm{FE}, \qquad \mathrm{AB} : \mathrm{EB} = \mathrm{DA} : \mathrm{FE} \ldots$$

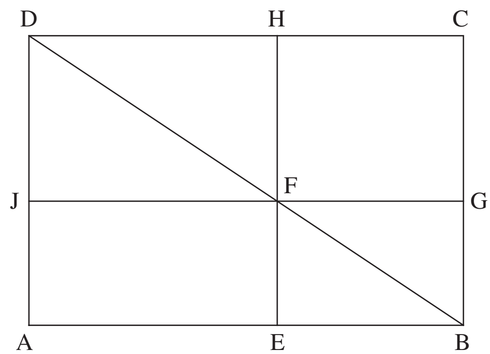


Figure 3 Chinese use of IOCP in determining "similarity" of right triangles

Of course, now that I understand what the Chinese did, rereading the other problems and their solutions make more sense. What I had previously considered inefficient procedures supported a use of in-out techniques. One of my favorite problems of this section, one which has also remained popular among algebra teachers for two thousand years, is the 16th exercise [**12**, p. 49]. When I first encountered it, I was perplexed by the given solution technique.

(Given) [*right triangle*] *gou 8 units long and gu 15 units. What is the largest circle that can be inscribed in this triangle?*

(Answer) *A circle with diameter of 6 units.*

(Method) *Find the length of xuan from gou and gu. The diameter will be the quotient of twice the product of gou and gu and the sum of gou, gu and hsien.*

In Chinese terminology the long side of a right triangle is called *gu*; the short side, *gou* and the hypotenuse, *xuan* [*hsien*]. In the explanatory diagrams of the Chinese solution process given in FIGURE 4 (A Proof Without Words?) I shall label the length of *gou*, a, the length of *gu*, b, and the length of *xuan*, c, and the diameter of the desired circle, d.

As shown in the top part of FIGURE 4, the Chinese first doubled the area of the given triangle by joining it to itself to form a rectangle of area ab. Then they dissected the pieces, selected two sets of them and arranged them into the rectangle shown in the bottom part of FIGURE 4. From this latter diagram, we see:

$$d \times (a + b + c) = 2ab,$$

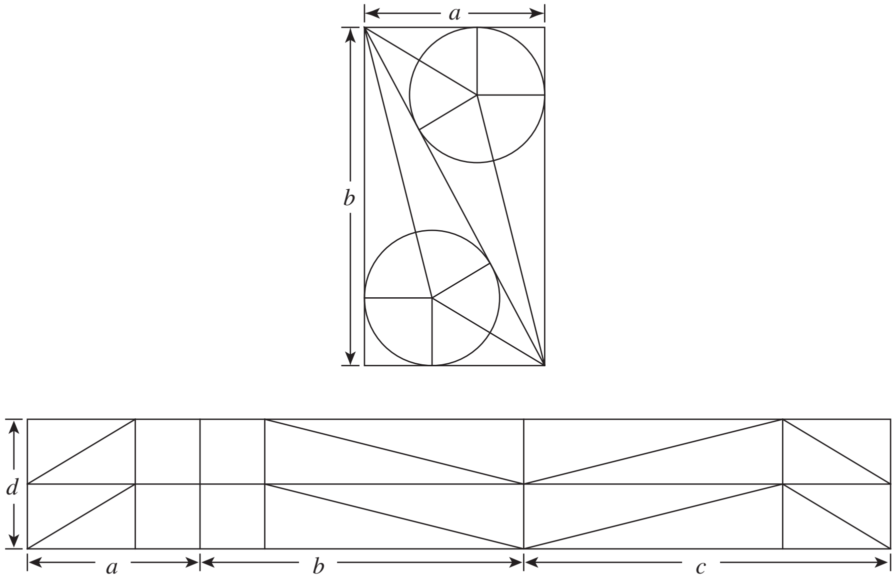


Figure 4 Problem 16

or

$$d = \frac{2ab}{a+b+c}.$$

And in this particular situation:

$$d = \frac{2 \times 8 \times 15}{8+15+17} = 2\left(\frac{120}{40}\right)$$

From my western perspective, I wondered why the author of the problem set it up to find the diameter, d, of the circle rather than the radius which would have resulted in a simpler diagram. Now I see, it was to accommodate a use of IOCP, conceiving of the rectangle of area ab divided by diagonal of length c allowed for the joining of complementary regions into smaller rectangles interpreted as products which resulted in the given solution formula. The solution for previous problem in this series is also obtained in the similar manner [**12**, p. 48].

Extending *Gou gu* employing the In-Out Principle

In the year 263, the scholar-official Liu Hui, wrote a commentary on the *Jiuzhang* correcting existing mistakes and providing justifications for much of the computational methodology. In particular, Liu expanded the *Gou-gu* chapter by considering distance determining situations where two distinct sightings are taken. The first problem demonstrating this situation involved a "Sea Island" [**1**]. See FIGURE 5.

The problem is as follows:

(Given) *Now for* [*the purpose*] *of looking at a sea island, erect two poles of the same height, 3 zhang* [*on the ground*], *the distance between the front and rear pole being 1000 bu. Assume that the rear pole is aligned with the front pole. Move away 123 bu from the front pole and observe the peak of the*

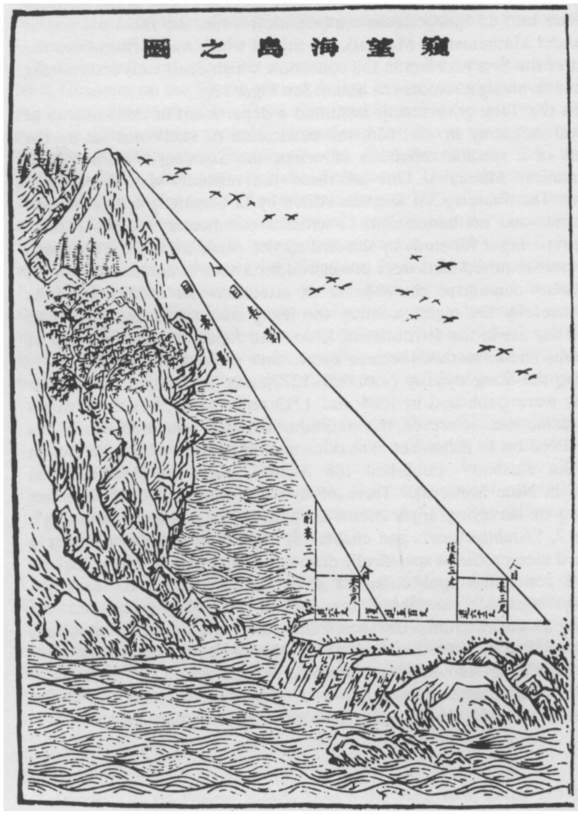


Figure 5 The Sea Island Problem

island from ground level; it is seen that the tip of the front pole coincides with the peak. Move 127 bu from the rear pole and observed the peak of the island from ground level again; the tip of the back pole also coincides with the peak. What is the height of the island and how far is it from the pole?

(Answer) *The height of the island is 4 li 55 bu. It is 102 li 150 bu from the* [*near*] *pole.*

(Method) *Multiply the distance between poles by the height of the pole, giving that shi. Take the difference in distance from the points of observation as the fa to divide* [*the shi*] *and add what is obtained to the height of the pole. The result is the height of the island. To find the distance of the island from the front pole, multiply* [*the distance of the*] *backward movement from the front pole by the distance between the poles, giving the shi. Take the difference in distance at the points of observation as the fa to divide there shi. The result is the distance of the island from the pole in li.*

[A *zhang* is 10 *chi*; therefore 3 *zhang* equal 5 *bu*. Referring to calculations on a traditional computing board, *fa* and *shi* are technical terms denoting positions on the board and indicate the divisor and dividend, respectively.]

Before reading the discussion of the Chinese solution strategy, the reader should try to obtain the solution using their own contemporary methods. The later comparisons may be surprising.

Using modern notation, the geometric situation is depicted in the diagram of FIGURE 6 where the distances AS, CN, SN, SB, and ND are given. Let PQ represent the height of the island and QS the distance to the first sighting pole, AS. Sightings to

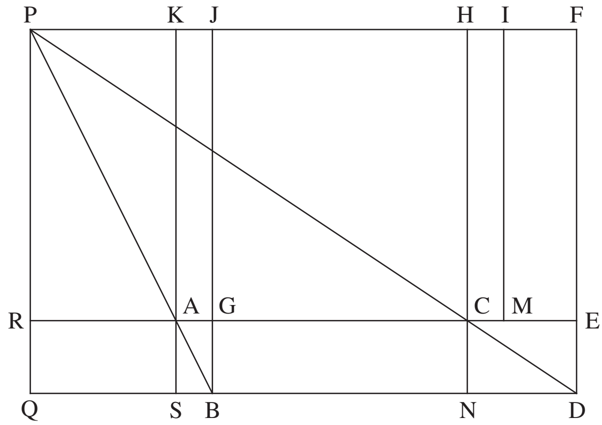


Figure 6 Construction for the Sea Island Problem

the peak of the island are taken at points B and D. Rectangles are constructed incorporating the knowns and unknowns. The auxiliary rectangle $\square$CI is constructed with CM = AG = SB.

Employing IOCP:

$$\text{area}(\square QC) = \text{area}(\square CF)$$
$$\text{area}(\square QA) = \text{area}(\square AJ) = \text{area}(\square CI)$$
$$\text{area}(\square SC) = \text{area}(\square QC) - \text{area}(\square QA).$$

From these we obtain

$$\text{area}(\square SC)) = \text{area}(\square CF) - \text{area}(\square CI) = \text{area}(\square MF)$$

so

$$\text{area}(\square SC) = \text{area}(\square MF)$$

and it follows that

$$(AS)(SN) = (PR)(ME) = PR(ND - CM),$$

but since CM = SB,

$$(AS)(SN) = PR(ND - SB).$$

Solving for PR:

$$PR = \frac{(AS)(SN)}{ND - SB},$$

and then the desired height is PQ = PR + RQ, and since RQ = AS, we have

$$PQ = \frac{(AS)(SN)}{ND - SB} + AS,$$

the given solution.

In a similar manner, the distance to the island is found to be

$$QS = \frac{(SB)(SN)}{ND - SB}.$$

Using the numbers from the problem (and converting the pole height 3 *zhang* to 5 *bu*) these equations give

$$\mathrm{PQ} = \frac{(5)(1000)}{(127-123)} + (5) = 1255\ bu = 3\ li\ 55 bu$$

for the height of the island, and

$$\mathrm{QS} = \frac{(123)(1000)}{(127-123)} = 30750\ bu = 102\ li\ 150\ bu$$

for its distance from the near pole, confirming the results obtained by Liu Hui.

Liu devised eight more such problems, each involving two observations. I found that the solutions given for all of them can be justified by IOCP. Wu Wenjun is also of the opinion that Liu obtained his results employing IOCP as was Yang Hui in his 13th century commentary on Liu's work [**15**, p. 69]. Eventually, this collection of problems became known as *Haidao suanjing* [The Sea Island Mathematical Manual], a separate Chinese mathematical classic [**11**]. In his recent examination of the Sea Island problem, Joseph Dauben also used IOCP to derive the solution [**4**, pp. 289–291].

Conclusions

In working on Chinese problems involving right triangles, it was so easy for me, with my western mathematical conditioning, to see the similarity of triangles as a method of solution. But still I had a nagging feeling that something wasn't quite right. If the Chinese recognize the similarity of right triangles, why didn't they recognize the concept of similarity in general?

Louis van Heé, a Belgian mathematician and sinologist working in the 1920s, made a translation of the Sea Island Mathematical Manual into French and analyzed its contents. From his studies, he concluded that the Chinese understood the concept of a similarity of triangles but their understanding was limited only to right triangles. Again, this conclusion seemed strange but could perhaps be justified in the fact that almost all the geometry problems the Chinese considered were practical applications involving surveying and distance observations, all depending on a use of right triangles. Chinese surveyors used sighting poles or gnomons and their observations and conclusions usually depended on pairs of measurements: the length of the pole and the length of its projection or "shadow" on the ground [**10**]. In such sighting situations, a series of right triangles is generated with the same base line and acute sighting angle. The surveyors and mathematicians correctly computed proportions between the sides of these triangles. Indeed these triangles are similar but the "principle of similarity" was not recognized or used by Chinese mathematicians. Historically, in no other instances of traditional Chinese mathematical work are found any references to similar polygons other than the right triangles just described. A Euclidean concept of similarity was not introduced into China until the 17th century (1607) when the first six books of Euclid as taken from Christopher Clavius's edition of the time were translated into Chinese by Jesuit missionaries [**2**].

I learned about IOCP soon after completing work on *Was Pythagoras Chinese?* but I did not make the necessary connection with the right-triangle problems. Further pondering of the seemingly inefficient Chinese approach to setting up the solution of problem 16, considered above revealed my faux pas—of course, the author of the solution formed the geometric data into a rectangle by considering two congruent triangles joined together. He then employed IOCP. My revised conclusion that the problems of the *Gou-gu* chapter of *Jiuzhang* were solved by applications of IOCP was further reinforced by the discovery that Liu Hui in his original commentary on this mathematical classic advised the readers to explore solutions using dissections: "Using diagrams on small paper, cut diagonally through intersections, rearrange them together, combining them to form particular shapes..." [**7**, p. 71].

Now duly enlightened, I humbly apologize to my Chinese mathematical predecessors for my intellectual arrogance and urge colleagues who may be undertaking similar historical mathematical explorations; "Be careful at what you may assume about a solution process. The ancients may not have done it our way, perhaps they even did it a better way!"

REFERENCES

1. Ang Tian Se and F. J. Swetz, A Chinese mathematical classic of the third century: The Sea Island Mathematical Manual of Liu Hui, *Historia Mathematica* **8** (1979) 218–220.
2. Christopher Clavius, *Euclides elementorum* libri xv, Rome (1574).
3. Christopher Cullen, *Astronomy and Mathematics in Ancient China: the Zhou bi suan jing*, Cambridge University Press, New York (1996).
4. Joseph W. Dauben, Chinese Mathematics, in *The Mathematics of Egypt, Mesopotamia, China, India and Islam: A Sourcebook*, Victor Katz, ed., Princeton University Press, Princeton (2007) 187–380.
5. T. Heath, *A History of Greek Mathematics*, 2 vols., 1921; also Dover Publications, New York, 1981.
6. Lam Lay Yong, Review: *Was Pythagoras Chinese? Historia Mathematica* **8** (1979) 18–220.
7. Li Yan and Du Shiran, *Chinese Mathematics: a Concise History*, translated by J. Crossley and A.Lun, Claredon Press, Oxford, 1987.
8. Shen Kangshen et al, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Oxford University Press, 1999.
9. D. E. Smith, *History of Mathematics*, 2 vols., 1923; also Dover Publications, New York, 1958.
10. Frank J. Swetz, Trigonometry Comes Out of the Shadows, in *Learn from the Masters*, F. Swetz et al., eds., The Mathematical Association of America, Washington, D.C. (1995) 57–73.
11. Frank J. Swetz, *The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China*, Pennsylvania State University Press, University Park, PA, 1922.
12. Frank. J. Swetz and T.I. Kao, *Was Pythagoras Chinese? An Examination of Right Triangle Theory in China*, Pennsylvania State University Press, University Park, PA, 1977.
13. Louis van Heé, Le classique de l'Île maritime: ouvrage chinos de IIIe siécle, in *Quellen und Studien zur Geschichte der Mathematik* **2** (1932) 225–258.
14. Louis van Heé, Le Hai-Tao suan-king, 20 Toug pao (1920) 51–60.
15. Wu Wenjun, The out-in complementary principle, *Ancient China's Technology and Science*, Chinese Academy of Science, Beijing (1983) 66–89.
16. Wu Wenjun, Churu xiangbu yuanli [Out-In-complementary Principle] in *Zhongguo gudai keji chengjiu* [Achievements of Ancient Chinese Science and Technology], Institute for History of Natural Sciences, Beijing (1978).
17. Yang Hui, *Xiangjie jiuzhang suanfa* [A Detailed Analysis of the Mathematical Methods in the Nine Chapters], Commercial Press, Shanghai (1936).

Summary Modern investigators of early Chinese mathematical classics have often attributed a recognition and application of the principles of geometric similarity to the authors and commentators of these works. In problem situations involving the use of sighting poles and the determination of a remote distance, Chinese mathematicians frequently employed proportionality relations involving the sides of relevant pairs of right triangles. In such situations the modern western observer sees similarity; however, the Chinese employed a concept now called "the In-and-Out-Complementary-Principle," IOCP. They did not use geometric similarity. This article identifies the issues of confusion and examines the concept and application of IOCP.

FRANK SWETZ is Professor Emeritus of Mathematics and Education at the Pennsylvania State University Harrisburg. At this institution he helped establish and chaired the Mathematical Sciences program. His research interests focus on societal impact on the learning and teaching of mathematics. This quest has taken him into studies about the history of mathematics, ethnomathematics and problem solving. A firm advocate of incorporating the history of mathematics into its teaching, he has published several books on this topic. The most recent effort in this series is *Mathematical Expeditions: Exploring Word Problems from Other Ages*, Johns Hopkins University Press, 2012. For recreation he enjoys conversation, traveling, and gardening.

Properties of Eisenstein Triples

RUSSELL A. GORDON
Whitman College
Walla Walla, WA 99362
gordon@whitman.edu

An Eisenstein triple (a, b, c) consists of three positive integers $a < c < b$ such that $a^2 - ab + b^2 = c^2$. By the law of cosines, these triples correspond to triangles that have integral side lengths and contain exactly one 60 degree angle (opposite the side with length c). Eisenstein triples have been discussed by Gilder [**8**] and in this MAGAZINE by Beauregard and Suryanarayan [**3**], and many interesting questions remain to be explored. In this paper we seek to determine Eisenstein triples (a, b, c) for which a given positive integer x appears as one of the sides a or b or as one of the quantities $c - a$ or $b - c$. It turns out that all of the resulting triples depend on the positive divisors of the integer $3x^2$, and that the numbers of such triples are related in some interesting and unexpected ways.

As you might imagine, problems and techniques appearing in a study of Pythagorean triples (positive integers that satisfy $a^2 + b^2 = c^2$) are also relevant to a study of positive integers that satisfy $a^2 - ab + b^2 = c^2$, as well as the related equation $a^2 + ab + b^2 = c^2$ involving a triangle with a 120 degree angle (see Section 4). In fact, these three equations are the only options for integer-sided triangles having a specified angle with a rational number of degrees [**14**, p. 41].

Before proceeding, we explain why we have chosen to use the term "Eisenstein triples." In a way that mirrors the relationship between Pythagorean triples and Gaussian integers (numbers of the form $m + n\sqrt{-1}$, with $m, n \in \mathbb{Z}$), Eisenstein triples are related to Eisenstein integers. These are numbers of the form $m + n\,\omega$, where m and n are integers and $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. We say more about this connection toward the end of the paper and discuss how Eisenstein integers can be used to completely determine which positive integers x can appear as c in an Eisenstein triple. The use of the term "Eisenstein triple" arises from this relationship and seems to first be used by Cuoco [**5**].

Elementary properties

In this section, we give some simple examples of Eisenstein triples and discuss several of their elementary properties.

Two simple Eisenstein triples are (3, 8, 7) and (5, 8, 7). As this pair illustrates (and a straightforward computation shows), (a, b, c) is an Eisenstein triple if and only if $(b - a, b, c)$ is an Eisenstein triple; see FIGURE 1. Another simple calculation shows that if (a, b, c) is an Eisenstein triple, then so is (ka, kb, kc) for any positive integer k. We often want to distinguish between Eisenstein triples such as (3, 8, 7) and (7, 15, 13), which represent qualitatively different (not similar) triangles, and triples such as (5, 8, 7) and (10, 16, 14), which represent triangles that are similar to each other. By a *primitive* Eisenstein triple, we mean an Eisenstein triple (a, b, c) for which the integers a, b, and c have no common positive divisor other than 1; that is, the integers a, b, and c are relatively prime. For example, the triples (5, 8, 7) and (7, 15, 13) are primitive whereas the triples (10, 16, 14) and (21, 45, 39) are not. Note that an Eisenstein triple is primitive if and only if any two integers in the triple are relatively

Math. Mag. **85** (2012) 12–25. doi:10.4169/math.mag.85.1.12.

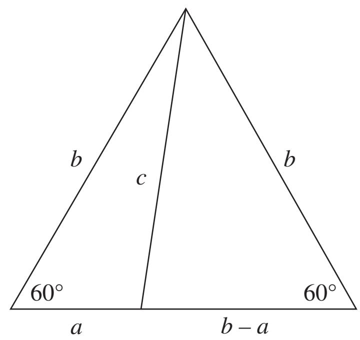


Figure 1 A pair of Eisenstein triples, (a, b, c) and $(b - a, b, c)$

prime. A consequence of this fact is that (a, b, c) is a primitive Eisenstein triple if and only if $(b - a, b, c)$ is also primitive.

Some further examples of primitive Eisenstein triples include (only one of each "natural" pair is recorded, the one with the smaller value for a)

$$(5, 21, 19),\ (7, 40, 37),\ (11, 35, 31),\ (11, 96, 91),\ (16, 55, 49),\ (31, 255, 241).$$

There are several different ways to generate Eisenstein triples and specific formulas appear below. Before proceeding, the reader might enjoy finding more examples or general formulas for Eisenstein triples.

It is known that the prime divisors of the hypotenuse c in a primitive Pythagorean triple must be of the form $4k + 1$. A quick perusal of the above examples reveals that the prime divisors of the term c in a primitive Eisenstein triple are primes of the form $6k + 1$. This fact follows most easily from the theory of quadratic residues. (For a discussion of quadratic residues see, for example, [**11**] or [**13**]. We note that the reader does not need to be familiar with quadratic residues to understand other results in this paper.)

THEOREM 1. *If (a, b, c) is a primitive Eisenstein triple, then c is neither a multiple of* 2 *nor* 3. *More generally, the only prime factors of c are primes of the form* $6k + 1$.

Proof. Since the triple (a, b, c) is primitive, the integers a and b cannot both be even. It follows that $c^2 = a^2 - ab + b^2$ is odd. This shows that c is not a multiple of 2. Suppose, by way of contradiction, that c is a multiple of 3. Since

$$(a + b)^2 = (a^2 - ab + b^2) + 3ab = c^2 + 3ab,$$

we see that $a + b$ is a multiple of 3. It follows that $3ab$ is a multiple of 9. But this means that 3 is a divisor of either a or b, a contradiction to the fact that (a, b, c) is a primitive Eisenstein triple. Now suppose that $p > 3$ is a prime that divides c. Writing $a^2 - ab + b^2 = c^2$ as

$$4c^2 = (2a - b)^2 + 3b^2$$

shows that -3 is a quadratic residue of p. Using the quadratic reciprocity theorem (see [**13**]), it is not difficult to show that p must be of the form $6k + 1$. ∎

The next theorem gives a further divisibility result by showing that the terms in any related pair of Eisenstein triples must be divisible by certain primes. The two smallest Eisenstein triples are the paired triples (3, 8, 7) and (5, 8, 7). The fact that the numbers 3, 5, 7, and 8 appear in these triples is no accident as multiples of these numbers are in every (a, b, c) and $(b - a, b, c)$ pair of Eisenstein triples. The reader may want to check this property (also mentioned in [**2, 3, 8**]) for the Eisenstein triples that have been presented thus far.

THEOREM 2. *If* (a, b, c) *is an Eisenstein triple, then*

(a) *one of the numbers* a, b, *or* $b - a$ *is divisible by* 8;

(b) *one of the numbers* a, b, *or* $b - a$ *is divisible by* 3;

(c) *one of the numbers* a, b, *or* $b - a$ *is divisible by* 5;

(d) *one of the numbers* a, b, c, *or* $b - a$ *is divisible by* 7.

Proof. We first prove part (a). Suppose that (a, b, c) is an Eisenstein triple and, without loss of generality, assume that (a, b, c) is primitive. By Theorem 1, the integer c must be odd. It follows easily that at least one of a or b is odd as well. Suppose that a is odd; the case in which b is odd is similar. Since both a and c are odd, it is easy to verify that $c^2 - a^2$ is a multiple of 8. Noting that $c^2 - a^2 = b(b - a)$, we conclude that either 8 divides b (if b is even) or 8 divides $b - a$ (if b is odd).

Parts (b), (c), and (d) can be proved by considering cases based on congruences modulo 3, 5, or 7, but this method of proof does become rather tedious. The website [**2**] includes these results but its proofs rely on a parametric form for Eisenstein triples. (We briefly mention parametric forms in a later section.) Here we prove part (d). Our proof uses Fermat's Little Theorem, which states that $n^{p-1} \equiv 1 \pmod{p}$ when p is a prime and n is not a multiple of p. The proofs of parts (b) and (c) are similar.

Once again, suppose that (a, b, c) is a primitive Eisenstein triple. By Fermat's Little Theorem, we know that 7 divides $ab(b^6 - a^6)$. Since

$$\begin{aligned} ab(b^6 - a^6) &= ab(b + a)(b^2 - ab + a^2)(b - a)(b^2 + ab + a^2) \\ &= abc^2(b - a)(b + a)(b^2 + ab + a^2), \end{aligned}$$

we find that 7 divides one of the terms a, b, c, $b - a$, $b + a$, or $b^2 + ab + a^2$. To prove the result, we need to rule out the possibilities that 7 divides $b + a$ or 7 divides $b^2 + ab + a^2$. In what follows, all congruences are modulo 7. Suppose that 7 divides $b + a$. It then follows that $b \equiv -a$ and (using the defining equation for an Eisenstein triple) $c^2 \equiv 3a^2$. Since (a, b, c) is primitive, these conditions show that none of the integers a, b, or c are multiples of 7. By Fermat's Little Theorem, we have

$$1 \equiv c^6 \equiv (3a^2)^3 \equiv 27a^6 \equiv 27 \equiv -1,$$

a contradiction. Now suppose that 7 divides $b^2 + ab + a^2$. Then 7 also divides

$$(b^2 + ab + a^2) - 7a^2 = (b - 2a)(b + 3a).$$

If 7 divides $b - 2a$, then $b \equiv 2a$ and $c^2 \equiv 3a^2$ so we reach the same contradiction as above. If 7 divides $b + 3a$, then $b \equiv -3a$ and $c^2 \equiv -a^2$, and Fermat's Little Theorem (which can be applied for the same reasons given above) yields

$$1 \equiv c^6 \equiv (-a^2)^3 \equiv -a^6 \equiv -1,$$

a contradiction. This completes the proof. ■

By definition, we know that $0 < a < c < b$ in any Eisenstein triple. The triangle inequality guarantees that $b < a + c$. Two other useful inequalities that are valid for Eisenstein triples are presented in the next theorem.

THEOREM 3. *If (a, b, c) is an Eisenstein triple, then $2c > 2b - a$ and $2c > a + b$.*

Proof. Given an Eisenstein triple (a, b, c), we find that

$$c^2 = a^2 - ab + b^2 = \left(b - \frac{a}{2}\right)^2 + \frac{3}{4}a^2 > \left(b - \frac{a}{2}\right)^2$$

and it readily follows that $2c > 2b - a$. (Dropping a perpendicular from the intersection point on the base to the left side of the triangle in FIGURE 1 and applying the Pythagorean theorem yields a geometric understanding of this equation.) Applying this inequality to the Eisenstein triple $(b - a, b, c)$ yields $2c > a + b$. ■

The second inequality in the statement of Theorem 3 can be written as $c-a > b-c$. This inequality shows that the gap between the first and third terms in an Eisenstein triple is greater than the gap between the second and third terms. In particular, there is no Eisenstein triple (a, b, c) for which $c - a = 1$.

Triples containing a given integer as first element

As mentioned in the introduction, given a positive integer x, we seek to determine Eisenstein triples (a, b, c) so that x appears as one of the quantities a, b, $c - a$, or $b - c$. As we will see, in each case the answers depend on the factors of $3x^2$. In this section we focus on the case $a = x$, for odd values of x. In the next section we extend the results to all x, and to the cases where x appears as b, $c - a$, or $b - c$.

We begin with a definition that will be useful when we wish to distinguish Eisenstein triples from primitive Eisenstein triples. We say that a positive integer i is a *unitary* divisor of a positive integer n if i divides n and the integers i and n/i are relatively prime. In this case n/i is also a unitary divisor of n. For example, note that 3 and 4 are unitary divisors of 12 but that 2 and 6 are not.

It is easy to see that the first element of an Eisenstein triple cannot be either 1 or 2; these facts follow from the implications

$$\begin{aligned} 1 - b + b^2 = c^2 \quad &\Rightarrow \quad (2c)^2 - (2b-1)^2 = 3 \quad \Rightarrow \quad c = 1 \text{ and } b = 1; \\ 4 - 2b + b^2 = c^2 \quad &\Rightarrow \quad c^2 - (b-1)^2 = 3 \quad \Rightarrow \quad c = 2 \text{ and } b = 2. \end{aligned}$$

The following theorem and its proof show how to find all of the Eisenstein triples that have a given odd integer greater than 1 as the first element.

THEOREM 4. *Let $x > 1$ be an odd positive integer. Then there is a one-to-one correspondence between the set of all Eisenstein triples that have x as first element and the set of all divisors i of $3x^2$ that satisfy $1 \le i < x$, and there is a one-to-one correspondence between the set of all primitive Eisenstein triples that have x as first element and the set of all unitary divisors i of $3x^2$ that satisfy $1 \le i < x$.*

Proof. Let $x > 1$ be a fixed odd positive integer, let E be the set of all Eisenstein triples that have x as first element, and let D be the set of all divisors i of $3x^2$ that satisfy $1 \le i < x$. Define a function $f\colon D \to E$ by

$$f(s) = \left(x, \frac{x}{2} + \frac{t-s}{4}, \frac{t+s}{4}\right),$$

where $t = 3x^2/s$. We will show that the function f is a bijection.

Write $f(s)$ as (a, b, c). We must show that $f(s) \in E$, which requires first that b and c be integers. Since x and s are odd integers, it is not difficult to verify that $3x^2 + s^2$ is a multiple of $4s$. It follows that

$$c = \frac{t+s}{4} = \frac{3x^2+s^2}{4s} \quad \text{and} \quad b = c + \frac{x-s}{2}$$

are integers. Noting that

$$\begin{aligned} c - a &= \frac{3x^2 - 4sx + s^2}{4s} = \frac{(3x-s)(x-s)}{4s}, \\ b - c &= \frac{x-s}{2}, \end{aligned}$$

we see that the conditions on s guarantee that $0 < a < c < b$. Finally, the defining equation for an Eisenstein triple follows from

$$\begin{aligned} b^2 - c^2 &= (b-c)(b+c) \\ &= \left(\frac{x-s}{2}\right)\left(\frac{x+t}{2}\right) \\ &= \frac{1}{4}\left(x^2 + (t-s)x - 3x^2\right) \\ &= x\left(-\frac{x}{2} + \frac{t-s}{4}\right) \\ &= a(b-a). \end{aligned}$$

Thus $f(s)$ is an Eisenstein triple with $a = x$.

Suppose that $f(s_1) = (a, b, c) = f(s_2)$ for integers s_1 and s_2 in D. Given the definition of f, we find that $s_1 = a - 2(b-c) = s_2$. It follows that f is an injection.

Now suppose that (a, b, c) is an Eisenstein triple for which $a = x$. Define integers s and t by

$$s = 2c - (2b - a) \quad \text{and} \quad t = 2c + (2b - a).$$

The inequality $1 \le s < x$ follows from Theorem 3 and we find that

$$\begin{aligned} st &= 4c^2 - (2b-a)^2 \\ &= (4a^2 - 4ab + 4b^2) - (4b^2 - 4ab + a^2) \\ &= 3a^2 = 3x^2. \end{aligned}$$

It follows that $s \in D$. The equations $x = a$,

$$\frac{x}{2} + \frac{t-s}{4} = \frac{a}{2} + \frac{2b-a}{2} = b,$$

and $(t+s)/4 = c$ reveal that $f(s) = (a, b, c)$, and we conclude that f is a surjection. Putting the pieces together, we see that the mapping f establishes a one-to-one correspondence between the sets E and D.

For the second part of the theorem, we begin by listing the equations

$$\begin{aligned} a &= x, & 3x^2 &= st, \\ 4b &= 2x + t - s, & s &= 2c - (2b-a), \\ 4c &= t + s, & t &= 2c + (2b-a), \end{aligned}$$

used in the definition of the bijection f. Recalling that x is odd, it is not difficult to verify that

(i) any prime p that divides both s and t also divides x and thus each of a, b, c;
(ii) any prime p that divides each of a, b, and c also divides both s and t.

These facts guarantee that divisors of $3x^2$ that are not unitary correspond to Eisenstein triples that are not primitive. This completes the proof. ■

To illustrate this result, suppose that $x = 35$. The factors of $3x^2$ are

$$1,\ 3,\ 5,\ 7,\ 15,\ 21,\ 25,\ 35,\ 49,\ 75, 105,\ 147,\ 175,\ 245,\ 525,\ 735,\ 1225,\ 3675.$$

The possible options for s are only 1, 3, 5, 7, 15, 21, and 25 and we obtain the triples

$$(35, 936, 919),\quad (35, 323, 307),\quad (35, 200, 185),\quad (35, 147, 133),$$
$$(35, 75, 65),\quad (35, 56, 49),\quad (35, 48, 43),$$

respectively. The reader should determine which triples are primitive and note that the corresponding values for s are unitary divisors of 3675.

In addition to finding Eisenstein triples that contain a given value of x, the function f used in the proof of Theorem 4 can be used to generate general formulas for Eisenstein triples. For example, for each positive integer n, the triple (a, b, c) defined by

$$a = 2n + 1;$$
$$b = 3n^2 + 4n + 1;$$
$$c = 3n^2 + 3n + 1;$$

is a primitive Eisenstein triple; it corresponds to $f(1)$ using the unitary divisor 1 of $3(2n + 1)^2$. Consequently, every odd positive integer greater than 1 is a first element of at least one primitive Eisenstein triple.

Triples containing a given integer

We now extend the results of the previous section to other locations for the integer x and to include the possibility of even integers. To do so concisely, it is helpful to introduce some notation. For each positive integer x, define sets of positive integers

$$D_x = \{\text{all positive divisors of } 3x^2\};$$
$$D_x^1 = \{\text{all divisors } i \text{ of } 3x^2 \text{ that satisfy } 1 \le i < x\};$$
$$D_x^2 = \{\text{all divisors } i \text{ of } 3x^2 \text{ that satisfy } x < i < 3x\};$$
$$D_x^3 = \{\text{all divisors } i \text{ of } 3x^2 \text{ that satisfy } 3x < i \le 3x^2\};$$

and sets of Eisenstein triples

$$E_x^1 = \{\text{all Eisenstein triples } (a, b, c) \text{ for which } x = a\};$$
$$E_x^2 = \{\text{all Eisenstein triples } (a, b, c) \text{ for which } x = b\};$$
$$E_x^{1.3} = \{\text{all Eisenstein triples } (a, b, c) \text{ for which } x = c - a\};$$
$$E_x^{2.3} = \{\text{all Eisenstein triples } (a, b, c) \text{ for which } x = b - c\}.$$

The notation is intended to help describe the sets. For instance, the superscript in $E_x^{1.3}$ indicates that x is the gap between the first and third terms in an Eisenstein triple. The superscripts 1, 2, and 3 on the D sets represent the factors of $3x^2$ that are in a first, second, or third group of factors when listed in increasing order. We use the prefix p on an E symbol to indicate the primitive Eisenstein triples of that type. For instance, the symbol pE_x^1 represents the collection of all primitive Eisenstein triples (a, b, c) for which $x = a$. Similarly, the prefix u on a D symbol indicates the relevant unitary divisors. For example, the symbol uD_x^2 represents the collection of all unitary divisors i of $3x^2$ that satisfy $x < i < 3x$.

The following theorem summarizes the relationships that exist between the E sets and the D sets for a given positive integer x. We use the symbols $A \sim B$ to indicate that there is a bijection between the sets A and B.

THEOREM 5. *Let x be a positive integer.*

(1) $E_x^{1.3} \sim D_x^3$ *and* $pE_x^{1.3} \sim uD_x^3$.
(2) $E_x^{2.3} \sim D_x$ *and* $pE_x^{2.3} \sim uD_x$.
(3) $E_x^1 \sim D_x^1$ *and* $pE_x^1 \sim uD_x^1$, *assuming x is an odd integer.*
(4) $E_x^2 \sim D_x^2$ *and* $pE_x^2 \sim uD_x^2$, *assuming x is an odd integer.*
(5) $E_x^1 \sim E_{x/2}^1$ *and* $pE_x^1 = \phi$, *assuming* $x/2$ *is an odd integer.*
(6) $E_x^2 \sim E_{x/2}^2$ *and* $pE_x^2 = \phi$, *assuming* $x/2$ *is an odd integer.*
(7) $E_x^1 \sim E_{x/4}^1$ *and* $pE_x^1 = \phi$, *assuming* $x/4$ *is an odd integer.*
(8) $E_x^2 \sim E_{x/4}^2$ *and* $pE_x^2 = \phi$, *assuming* $x/4$ *is an odd integer.*
(9) $E_x^1 \sim D_{x/4}^1$ *and* $pE_x^1 \sim uD_{x/4}^1$, *assuming* $x/8$ *is an integer.*
(10) $E_x^2 \sim D_{x/4}^2$ *and* $pE_x^2 \sim uD_{x/4}^2$, *assuming* $x/8$ *is an integer.*

Proof. Part (3) is a restatement of Theorem 4 using our new notation. To prove part (5), we simply need to show that the obvious correspondence $(a, b, c) \to (2a, 2b, 2c)$ between the sets $E_{x/2}^1$ and E_x^1 is onto. In other words, we need to show that if (a, b, c) belongs to E_x^1, then the integers a, b, and c are even. By Theorem 2, we know that either a, b, or $b - a$ is a multiple of 8. The conditions on $a = x$ rule out the possibility that a is a multiple of 8. We thus find that either $b = 8n$ or $b = 8n + a$ for some positive integer n. In particular, the integer b is even and this implies that c is even as well.

The proofs of parts (6), (7), and (8) are almost identical to the proof of part (5). The proofs of parts (1), (2), (4), (9), and (10) are similar to the proof of part (3) that can be found in Section 2. The appropriate bijections for each correspondence are

(1) $g\colon D_x^3 \to E_x^{1.3}$ defined by $g(t) = (s + t - 4x, t - 2x, s + t - 3x)$, where $s = 3x^2/t$;
(2) $G\colon D_x \to E_x^{2.3}$ defined by $G(t) = (2x + t, 4x + s + t, 3x + s + t)$, where $s = 3x^2/t$;
(4) $F\colon D_x^2 \to E_x^2$ defined by $F(s) = \left(\frac{x}{2} + \frac{t-s}{4}, x, \frac{t+s}{4}\right)$, where $t = 3x^2/s$;
(9) $h\colon D_{x/4}^1 \to E_x^1$ defined by $h(s) = \left(x, \frac{x}{2} + t - s, t + s\right)$, where $t = 3(x/4)^2/s$;
(10) $H\colon D_{x/4}^2 \to E_x^2$ defined by $H(s) = \left(\frac{x}{2} + t - s, x, t + s\right)$, where $t = 3(x/4)^2/s$.

We leave the details to the reader. However, it is worth noting that each proof does involve a few nontrivial adaptations to the proof of part (3). ■

To illustrate the functions given in the proof, suppose that $x = 35$ and factor $3x^2$ to determine that

$$D_{35} = \{1, 3, 5, 7, 15, 21, 25, 35, 49, 75, 105, 147, 175, 245, 525, 735, 1225, 3675\};$$
$$D^1_{35} = \{1, 3, 5, 7, 15, 21, 25\};$$
$$D^2_{35} = \{49, 75\};$$
$$D^3_{35} = \{147, 175, 245, 525, 735, 1225, 3675\}.$$

We find that (for example)

$$g(175) = (56, 105, 91); \qquad f(5) = (35, 200, 185);$$
$$G(49) = (119, 264, 229); \qquad F(75) = (11, 35, 31).$$

The reader can determine all of the Eisenstein triples associated with $x = 35$ and note that unitary divisors of $3 \cdot 35^2$ do indeed correspond to primitive Eisenstein triples.

As a specific illustration for results (5) through (8), to find the sets E^1_{70} and E^1_{140}, we simply need to take the elements of E^1_{35} and multiply each of the integers in the triple by 2 or 4, respectively. Consequently, none of these Eisenstein triples is primitive.

Finally, to illustrate the results of parts (9) and (10), suppose that $x = 96$. Then $96/4 = 24$ and

$$D^1_{24} = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18\}, \quad D^2_{24} = \{27, 32, 36, 48, 54, 64\}.$$

Using the functions defined above, we find that (for example)

$$h(1) = (96, 1775, 1729) \quad \text{and} \quad H(36) = (60, 96, 84).$$

Note that the first triple is primitive (with unitary divisor 1) and that the second triple is not primitive (36 is not a unitary divisor of $3 \cdot 24^2$).

We now explore and illustrate some of the consequences of Theorem 5, beginning with a sharp contrast between Eisenstein triples and Pythagorean triples. For the latter (assuming $a < b < c$) there are an infinite number of Pythagorean triples of each of the forms $(a, a + x, c)$ and $(a, b, b + x)$ for each positive integer x; see [**6, 7, 12, 16, 17**] for several of the many resources on this topic. As part (a) of the next theorem indicates, the corresponding sets of Eisenstein triples are finite.

THEOREM 6.

(a) *For each positive integer x, the sets $E^{1.3}_x$ and $E^{2.3}_x$ are finite. That is, there are only finitely many Eisenstein triples of the form $(a, b, a + x)$ or $(a, c + x, c)$.*

(b) *For each positive integer x, the set $pE^{2.3}_x$ is nonempty, and for each positive integer $x > 1$, the set $pE^{1.3}_x$ is nonempty.*

Proof. Part (a) follows immediately from the one-to-one correspondences between triples and divisors, while part (b) follows from the fact that 1 is always a unitary divisor of $3x^2$. ■

Let $\tau(x)$ represent the number of positive divisors of the positive integer x and let $|A|$ denote the number of elements in the finite set A. For the record, if $x = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where the p_i's are distinct primes and the e_i's are positive integers, then $\tau(x) = \prod_{i=1}^n (e_i + 1)$. In particular, when x is a positive integer, the number $\tau(3x^2)$ is even. Furthermore, it is easy to see that $|D^1_x| = |D^3_x|$ and

$$|D^1_x| + |D^2_x| + |D^3_x| + 2 = |D_x| = \tau(3x^2).$$

For the $x = 35$ case considered earlier, this equation becomes $7 + 2 + 7 + 2 = 18$. The isolated number 2 that appears in this formula is needed to account for the factors x and $3x$ that do not appear in the sets D_x^1, D_x^2, and D_x^3. Using the functions defined previously, these factors yield the degenerate triples $(0, x, x)$ and (x, x, x), which we have not included as Eisenstein triples.

Given the one-to-one correspondences listed in Theorem 5 and the relationships between the sizes of the D sets, we obtain the following result. The simple details of the proof are omitted.

THEOREM 7. *The following relationships between the sizes of the E sets are valid.*

(a) $|E_x^1| = |E_x^{1.3}|$ *when x is an odd positive integer. That is, for x odd, there are as many Eisenstein triples of the form (x, b, c) as of the form $(a, b, a + x)$.*

(b) $|E_x^{2.3}| = \tau(3x^2)$ *when x is a positive integer.*

(c) $|E_x^1| + |E_x^2| + |E_x^{1.3}| + 2 = |E_x^{2.3}|$ *when x is an odd positive integer.*

(d) $2|E_x^1| + |E_x^2| = \tau(3x^2) - 2$ *when x is an odd positive integer.*

(e) $|E_x^1| + |E_x^2| + |E_{x/2}^{1.3}| + 2 = |E_{x/2}^{2.3}|$ *when $x/2$ is an odd positive integer.*

(f) $|E_x^1| + |E_x^2| + |E_{x/4}^{1.3}| + 2 = |E_{x/4}^{2.3}|$ *when $x/4$ is a positive integer.*

As we have done for part (a), it might be helpful to translate these symbolic statements into words. For example, part (c) says that the number of times that a given odd positive integer x appears as either a first element, a second element, or a gap between the first and third elements of an Eisenstein triple is two less than the number of times that x appears as the gap between the second and third elements of an Eisenstein triple.

Similarly, focusing on the unitary divisors of $3x^2$ and using our previous correspondences yield the following two results.

THEOREM 8. *Let $x > 1$ be an odd positive integer and let $n \geq 0$ be the number of distinct primes that are larger than 3 that appear in the prime factorization of x. Then*

(a) $|pE_x^1| = |pE_x^{1.3}|$*;*

(b) $|pE_x^1| + |pE_x^2| + |pE_x^{1.3}| = |pE_x^{2.3}| = 2^{n+1}$*;*

(c) $2|pE_x^1| + |pE_x^2| = 2^{n+1}$.

Proof. The fact that $|pE_x^1| = |pE_x^{1.3}|$ is clear since the unitary divisors of x occur in pairs and each unitary divisor in D_x^1 corresponds to a unitary divisor in D_x^3. If m is the number of distinct primes that appear in the factorization of a positive integer $y > 1$, then the number of unitary divisors of y is 2^m. To prove part (b), we note that the prime factorization of $3x^2$ contains $n + 1$ distinct primes. Furthermore (since $x > 1$), the factors x and $3x$ are not unitary divisors of $3x^2$. The equality now follows from the one-to-one correspondences established earlier. Part (c) is simply a different way of expressing the equations that appear in parts (a) and (b). ∎

THEOREM 9. *Let x be a positive multiple of 8 and let $n \geq 0$ be the number of distinct primes that are larger than 3 that appear in the prime factorization of x. Then*

(a) $|pE_x^1| = |pE_{x/4}^{1.3}|$*;*

(b) $|pE_x^1| + |pE_x^2| + |pE_{x/4}^{1.3}| = |pE_{x/4}^{2.3}| = 2^{n+2}$*;*

(c) $2|pE_x^1| + |pE_x^2| = 2^{n+2}$.

Proof. The proof is similar to the proof of the previous theorem; the details are left to the reader. ∎

The reader is invited to choose several values for x and find all of the Eisenstein triples that belong to the various sets. For the record, it is possible to write simple programs that find all the Eisenstein triples of a certain type and, more importantly, the programs can be written so as NOT to depend on factoring $3x^2$. (The author generated lots of data using simple exhaustive search programs in Maple.) Counting the number of triples in each set provides independent verification (not proofs, of course) of the formulas that appear in the last few theorems.

We end this section with a brief comment on the relationship between our formulas for primitive triples belonging to E_x^1 and E_x^2 and the formulas found in [**3**]. Suppose that $x > 3$ is an odd integer that is not a multiple of 3. If $x = pq$, where p and q are unitary divisors of x, then p^2 and $3q^2$ are unitary divisors of $3x^2$. Assuming that $s = p^2 < x$, the bijection f defined in the proof of Theorem 4 generates the triple

$$\left(x, \frac{x}{2} + \frac{3q^2 - p^2}{4}, \frac{3q^2 + p^2}{4}\right) = \left(pq, \frac{p^2 - 2pq - 3q^2}{-4}, \frac{p^2 + 3q^2}{4}\right),$$

which corresponds to equation (9) in [**3**]. Similar correspondences exist between our other formulas for primitive triples and those found in [**3**].

Triangles with a 120° angle

As pointed out in [**2**] and [**3**], we can just as easily consider triangles with an angle of 120° instead of 60°. We say that (a, b, c) is a 120° triple if $0 < a < b < c$ and $a^2 + ab + b^2 = c^2$. The corresponding triangle has a 120° angle opposite the side of length c. As indicated in FIGURE 2, there is a simple relationship between triangles associated with 120° triples and those associated with Eisenstein triples.

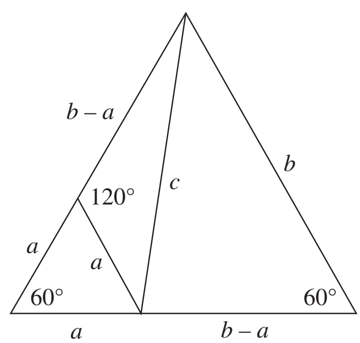


Figure 2 A relationship between 60° and 120° triangles

The figure shows that, for example, the Eisenstein triple (3, 8, 7) generates the 120° triple (3, 5, 7). In fact, the figure provides a geometric proof of the following general result; the algebraic details are left to the reader.

THEOREM 10. *Suppose that a, b, and c are positive integers.*

(a) *If* (a, b, c) *is an Eisenstein triple with* $a < b - a$, *then* $(a, b - a, c)$ *is a* 120° *triple.*

(b) *If* (a, b, c) *is a* 120° *triple, then* $(a, a + b, c)$ *and* $(b, a + b, c)$ *are Eisenstein triples.*

This connection between 120° triples and Eisenstein triples makes it possible to use solutions to Eisenstein triples problems to answer questions concerning 120° triples. As an indication of the sorts of results that are possible, we use the notation O_x^1 to represent 120° triples of the form (x, b, c) and so on. (The letter O refers to the fact that 120° triples correspond to triangles with an obtuse angle.) It can then be shown that for each odd positive integer $x > 1$, the correspondence $O_x^1 \cup O_x^2 \sim E_x^1$ is valid. In fact, the corresponding divisors s of $3x^2$ that generate elements of O_x^1 and O_x^2 satisfy

$$1 \leq s < \left(\sqrt{12} - 3\right) x \quad \text{and} \quad \left(\sqrt{12} - 3\right) x < s < x,$$

respectively. We leave such questions and solutions as a project for the reader. However, we would like to point out one clear distinction between these two types of triples. Given a positive integer x, the number of 120° triples of the form $(a, a + x, c)$ is infinite, that is, there are an infinite number of 120° triples for which the gap between the first two terms has a given value. (To phrase this another way, the number of Eisenstein triple pairs (a, b, c), $(b - a, b, c)$ for which $b - 2a$ is a given positive integer x is infinite.) One way to see this is to write $a^2 + ab + b^2 = c^2$ as

$$(b - a)^2 = (2c)^2 - 3(b + a)^2.$$

If we are interested in 120° triples for which $b - a = 1$, then we need to find solutions to the Pell equation $x^2 - 3y^2 = 1$ for which the integer x is even. (See [**13**] for one of many references for Pell equations.) Omitting the details, set $a_0 = 0$, $b_0 = 1$, and $c_0 = 1$, and for each positive integer n, define

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 4 & 3 & 4 \\ 3 & 4 & 4 \\ 6 & 6 & 7 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix}.$$

Then $\{(a_n, b_n, c_n)\}$ forms a sequence of 120° triples such that $b_n - a_n = 1$. The first few triples in this sequence are

$$(7, 8, 13), \quad (104, 105, 181), \quad (1455, 1456, 2521), \quad \text{and} \quad (20272, 20273, 35113).$$

The reader may consult [**2**] for a different approach that results in the same transition matrix that converts one 120° triple into another.

Connection to Eisenstein integers

We now turn to a brief consideration of the third term of an Eisenstein triple and thus clarify the connection between these triples and Eisenstein integers. As mentioned earlier in the paper, Eisenstein integers are numbers of the form $m + n\,\omega$, where m and n are integers and $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Note that $\omega^3 = 1$ and $\omega^2 = -1 - \omega = \overline{\omega}$, where the bar refers to complex conjugation. Eisenstein integers include all of the ordinary integers (set $n = 0$) and they behave like ordinary integers in many ways. In particular, there is a notion of divisibility for Eisenstein integers and this leads to the concept of Eisenstein primes. (In addition to Internet sites, the reader may consult [**4**] for a discussion of Eisenstein integers or [**13**] for a more general discussion of algebraic integers.) In the same way in which primes of the form $4k + 1$ are not Gaussian primes, it turns out that primes of the form $6k + 1$ are not Eisenstein primes. For instance, we have the following factorizations in the Gaussian integers (on the left) and in the Eisenstein integers:

$$5 = (2 + i)(2 + \overline{i}); \qquad 7 = (3 + \omega)(3 + \overline{\omega});$$

$$13 = (3 + 2i)(3 + 2\overline{i}); \qquad 13 = (4 + \omega)(4 + \overline{\omega}).$$

The parallels between Eisenstein integers and primitive Eisenstein triples and between Gaussian integers and primitive Pythagorean triples are quite strong and lead to the following result. We omit the details of the proof since they would take us too far astray. For corresponding proofs for Pythagorean triples with a given hypotenuse, see (among others) [**1, 7, 16**].

THEOREM 11. *If* $c = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$*, where each* p_i *is a prime of the form* $6k+1$ *and each* e_i *is a positive integer, then* c *appears as the third term in* $\tau(c^2) - 1$ *Eisenstein triples and* 2^n *of these Eisenstein triples are primitive.*

In keeping with our earlier notation, for each positive integer x, let E_x^3 denote the collection of all Eisenstein triples (a, b, c) for which $x = c$. We then have the following somewhat intriguing result.

THEOREM 12. *If* $x = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$*, where each* p_i *is a prime of the form* $6k+1$ *and each* e_i *is a positive integer, then*

$$2|E_x^1| + |E_x^2| = 2|E_x^3|.$$

That is, twice the number of triples having x *as a first element plus the number of triples having* x *as second element gives twice the number of triples having* x *as a third element.*

Proof. Combining the results of Theorem 11 and part (d) of Theorem 7, we find that

$$2|E_x^1| + |E_x^2| = \tau(3x^2) - 2 = 2\tau(x^2) - 2 = 2|E_x^3|.$$

Note that the equation $\tau(3x^2) = 2\tau(x^2)$ is valid since x is not a multiple of 3. ■

To illustrate this theorem, let $x = 91 = 7 \cdot 13$. Using the bijections given earlier to find the sets E_x^1 and E_x^2, we obtain

$$\begin{aligned} E_x^1 &= \{(91, 160, 139),\ (91, 195, 169),\ (91, 336, 301),\ (91, 520, 481), \\ &\qquad (91, 931, 889),\ (91, 2115, 2071),\ (91, 6256, 6211)\}; \\ E_x^2 &= \{(40, 91, 79),\ (51, 91, 79)\}; \\ E_x^3 &= \{(11, 96, 91),\ (85, 96, 91),\ (19, 99, 91),\ (80, 99, 91),\ (39, 104, 91), \\ &\qquad (65, 104, 91),\ (49, 105, 91),\ (56, 105, 91)\}. \end{aligned}$$

The interested reader can verify (using various methods) the triples that belong to the set E_x^3.

Parametric equations for triples

It is well known [**15**] that Pythagorean triples correspond to points on the unit circle that have rational coordinates. Using this idea or other simple ideas from number theory, we find that the positive integers a, b, and c with a an even integer form a primitive Pythagorean triple if and only if there exist relatively prime positive integers s and t such that $s < t$, $t - s$ is not a multiple of 2, and

$$a = 2st, \qquad b = t^2 - s^2, \qquad c = t^2 + s^2.$$

These parametric equations for Pythagorean triples have been known and used for centuries.

There are similar parametric equations for Eisenstein triples but there are some complicating features, partly due to the fact that these triples occur in pairs. One approach is to use the fact that an Eisenstein triple (a, b, c) corresponds to a point with rational coordinates on the ellipse given by the equation $x^2 - xy + y^2 = 1$; see [**10**] for this approach. As with Pythagorean triples, there are other ways to obtain parametric equations for Eisenstein triples (see [**8**] for another option). For our purposes, we note that (leaving the elementary proof to the reader) if (a, b, c) is a primitive Eisenstein triple, then exactly one of the numbers $a + b + c$ or $b - a + b + c$ is a multiple of 3. We then obtain the following result.

THEOREM 13. *The positive integers a, b, and c form a pair of primitive Eisenstein triples* (a, b, c) *and* $(b - a, b, c)$ *with* $a + b + c$ *not a multiple of* 3 *if and only if there exist relatively prime positive integers s and t such that* $s < t$, $t - s$ *is not a multiple of* 3, *and*

$$a = s^2 + 2st, \qquad b = t^2 + 2st, \qquad c = t^2 + st + s^2.$$

Proof. The details of the proof are left to the reader. One option is to consult [**8**] and another is to modify the proof found in the appendix of [**9**]. ■

It is possible to use the parametric form of Eisenstein triples to tackle some of the problems discussed earlier in the paper; see [**2**] for this approach to the divisibility results and [**3**] for some partial solutions to the number of triples containing a given integer. However, the arguments from this perspective are not as straightforward as one would hope. Once again, the interested reader is invited to pursue these lines of thought.

Conclusion

There are many avenues to explore concerning Eisenstein triples. We have traveled down a few of them in this paper and merely pointed the way toward several others, but we have by no means exhausted the possibilities. For example, it is possible to consider the perimeters of triangles associated with Eisenstein triples and 120° triples; see [**9**]. Some of the references listed below indicate other options. We conclude this paper by mentioning one further problem that is suitable for students with a minimal background in number theory. Given a positive integer x, determine the sizes of the sets E_x^1, E_x^2, $E_x^{1.3}$, and $E_x^{2.3}$. The sizes of the sets typically depend on the prime factorization of x. Two such results are

I. If $p > 3$ is a prime and n is a positive integer, then $|E_{p^n}^1| = 2n$, $|E_{p^n}^2| = 0$, and $|pE_{p^n}^1| = 2$.

II. If p and q are primes with $q > p > 3$, then

$$|E_{pq}^1| = 7, \quad |E_{pq}^2| = 2, \quad \text{and} \quad |pE_{pq}^1| = 3, \quad |pE_{pq}^2| = 2, \quad \text{when } q < 3p;$$
$$|E_{pq}^1| = 8, \quad |E_{pq}^2| = 0, \quad \text{and} \quad |pE_{pq}^1| = 4, \quad |pE_{pq}^2| = 0, \quad \text{when } q > 3p.$$

One can find many results of this nature, although the results become rather messy when more than two distinct primes are involved.

REFERENCES

1. A. Ayoub, Counting primitive Pythagorean triples, *Mathematics & Computer Education* **39** (2005) 37–41. http://dx.doi.org/10.2307/2691222.
2. F. Barnes, "Pythagorean Triples, etc." available at http://pythag.net/
3. R. Beauregard and E. R. Suryanarayan, Integral triangles, *Math. Magazine* **72** (1999) 287–294.
4. J. Conway and R. Guy, *The book of numbers*, Springer-Verlag, New York, 1996.
5. A. Cuoco, Meta-Problems in Mathematics, *College Math. J.* **31** (2000) 373–378. http://dx.doi.org/10.2307/2687452.
6. L. Dickson, *History of the theory of numbers*, Vol. 2, Carnegie Institution of Washington, Washington, 1920.
7. A. Fassler, Multiple Pythagorean number triples, *Amer. Math. Monthly* **98** (1991) 505–517. http://dx.doi.org/10.2307/2324870.
8. J. Gilder, Integer-sided triangles with a 60° angle, *Math Gazette* **66** (1982) 261–266. http://dx.doi.org/10.2307/3615511.
9. R. Gordon and J. Wells, On the perimeter of integral triangles, *Int. J. Pure Appl. Math.* **64** (2010) 541–565.
10. P. K. Hotchkiss, It's perfectly rational, *College Math. J.* **33** (2002) 113–117. http://dx.doi.org/10.2307/1558993.
11. C. Long, *Elementary introduction to number theory*, 2nd ed. D. C. Heath, 1972.
12. D. McCullough and E. Wade, Recursive enumeration of Pythagorean triples, *College Math. J.* **34** (2003) 107–111. http://dx.doi.org/10.2307/3595782.
13. I. Niven and H. Zuckerman, *An introduction to the theory of numbers*, 3rd ed., Wiley, 1972.
14. I. Niven, *Irrational Numbers*, The Carus Mathematical Monographs, No. 11, MAA, Washington, DC, 1956.
15. J. Rotman, *Journey into mathematics; an introduction to proofs*, Prentice Hall, 1998.
16. E. S. Rowland, "Pythagorean Triples," available at http://www.math.rutgers.edu/~erowland/pythagoreantriples.html
17. D. Wegener, Primitive Pythagorean triples with sum or difference of legs equal to a prime, *Fibonacci Quart.* **13** (1975) 263–277.

Summary An Eisenstein triple (a, b, c) consists of three positive integers $a < c < b$ such that $a^2 - ab + b^2 = c^2$. These triples share a number of properties with the more familiar Pythagorean triples, but there are some noticeable differences. In addition, new approaches are sometimes required to discover and prove these properties. This paper presents an introduction to Eisenstein triples and considers some of their elementary properties. One of the primary goals is to determine Eisenstein triples (a, b, c) for which a given positive integer x appears as one of the quantities a, b, $c - a$, or $b - c$. It turns out that all of the resulting triples depend on the positive divisors of the integer $3x^2$ and that the numbers of such triples satisfy some interesting and unexpected equations.

RUSSELL A. GORDON received his Ph.D. from the University of Illinois in 1987, writing his dissertation under the influence of Jerry Uhl. Since then he has been teaching at Whitman College and pursuing various mathematical ideas, including writing a graduate textbook on nonabsolute integration and a textbook for elementary real analysis. He enjoys eating his spouse's wonderful vegetarian cooking (for which doing the dishes is a small price to pay), watching movies with his teenage son, and hiking in the local mountains.

NOTES

The Proof Is in the Pizza

GREG N. FREDERICKSON
Department of Computer Science
Purdue University, West Lafayette, Indiana 47907
gnf@cs.purdue.edu

Of dissection, proof, and pizza

A geometric dissection is a cutting of one or more geometric figures into finitely many pieces that can be rearranged to form one or more other geometric figures. As visual demonstrations of relationships such as the Pythagorean theorem, dissections have had a surprisingly rich history, reaching back to Islamic mathematicians a millennium ago and Greek mathematicians more than two millennia ago [**3**]. Other examples of mathematical relationships illustrated by dissections include identities of sums of squares and of sums of cubes of natural numbers [**3, 4, 5**].

Most published dissections are of either polygons or polyhedra. However there are a few dissections of curved figures [**3, 9**]. They are challenging to create, because a convex curved boundary in an original figure persists in a target figure unless there is

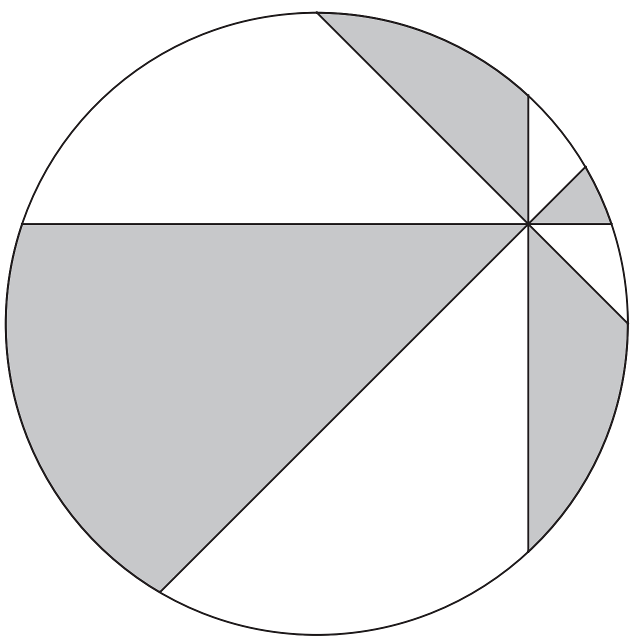


Figure 1 Upton's two sets of slices of equal total area

Math. Mag. **85** (2012) 26–33. doi:10.4169/math.mag.85.1.26. © Mathematical Association of America

a matching concave curved boundary in the original figure to offset it. It is not easy to identify curved figures that have this property and are also interesting objects for dissection.

A remarkable example arises when the pieces are created from a circular disk that is divided by chords, all of which pass through a single point within the disk at equally spaced angles. This sort of partition is typical of the way circular pizzas are cut, recognizing that whoever cuts the pizzas may not be careful to have the intersection of the chords be at the center of the disk.

When this *chordal intersection point* is at the center of the disk, then all resulting slices are congruent. However, when this point does not coincide with the center, then no two slices of opposite shading are congruent, unless one of the chords is a diameter. A half century ago, Leslie J. Upton, of suburban Toronto, in Canada, noticed a curious fact that holds even when no two slices are congruent [**12**]: If there are four chordal cuts, then the resulting eight slices can be partitioned into two sets, alternately taking one slice in one set and one slice in the other set, as in FIGURE 1, such that the total area of slices in one set equals the total area in the other. Michael Goldberg observed that whenever there are $k \geq 4$ chords and k is even, then the total area of one set of slices equals the total area of the other [**6**]. This is the so-called "pizza theorem," one of a host of related results [**11**].

A generation later, Larry Carter and Stan Wagon found a lovely dissection proof for Upton's original case of four chords [**2**]. They cut the four slices in one set into eight pieces that assemble to make the four slices in the other set. A variant [**8**] of their dissection is shown in FIGURE 2, where we see all eight slices together. Each piece appears in a white slice and again in a shaded slice. All angles at the intersections of line segments are integral multiples of 45°. The addition of a few line segments to the diagram makes it easy to see that the dissection is correct [**8**]. Soon after Carter and Wagon discovered their dissection, Allen Schwenk, at Western Michigan University, discovered the related dissection in FIGURE 3 for the case of 6 chords.

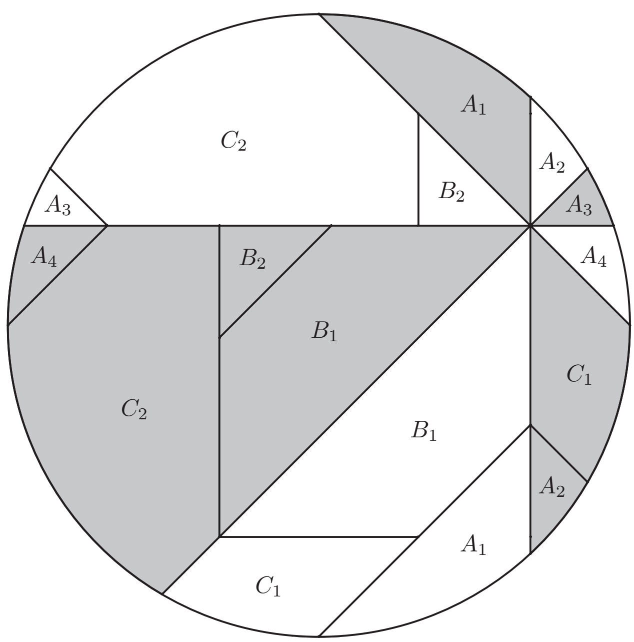


Figure 2 A variant of the Carter-Wagon dissection for 4 chords

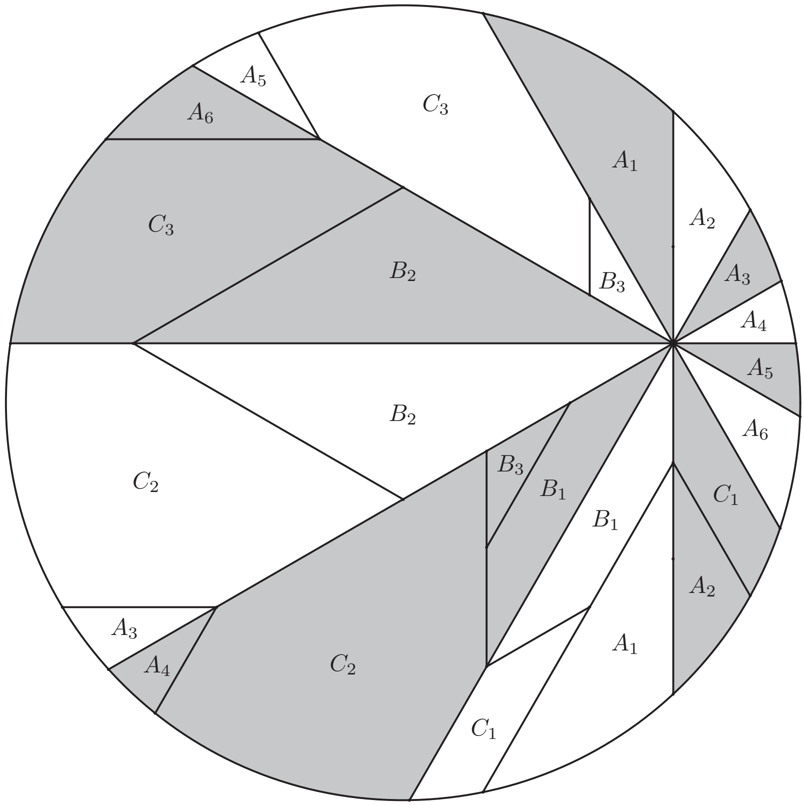


Figure 3 Allen Schwenk's dissection for 6 chords

Unlike previous dissections of curved figures, these are noteworthy because they are of a set of curved figures to a different set of curved figures. In this article, we extend these two dissections, by Carter and Wagon and by Schwenk, to a general dissection method that gives a proof of the pizza theorem for every even $k \geq 4$.

Franchising the recipe

We generalize the two preceding dissections to handle all members of the family in which we have k equally-spaced chords where $k \geq 4$ is even. Our construction, consistent with FIGURES 2 and 3, is given below for the case in which m_0, the common intersection point of the chords, falls inside and not on the boundary of the disk. (When m_0 falls on the boundary, our construction is still valid if we allow some of the pieces to have area 0.)

1. For each of the k chords ℓ_i, identify its *mirror point* m_i, which is the mirror image of m_0 across the midpoint of ℓ_i.
2. Rotate and possibly flip the disk so that there is a vertical chord, which we call ℓ_1, which has its mirror point m_1 directly beneath m_0, with the chords indexed in clockwise order, so that m_k is above and to the left of m_0.
3. Leave as uncut the k slices starting with the top one to the left of ℓ_1 and proceeding clockwise, giving pieces A_1 through A_k. For $i = 1, 2, \ldots, k/2$, reflect pieces A_{2i-1} and A_{2i} across the diameter that is perpendicular to the chord ℓ_{2i-1}, giving k mirror image pieces.

4. For $i = 2, \ldots, k/2 - 1$, cut a matching pair of isosceles triangles B_i, with bases spanning from m_0 to m_{2i} and base angles of $180°/k$.
5. Cut a similar isosceles triangle $B_{k/2}$ to the left of l_k with its base spanning from m_k to m_0. Take a similar isosceles triangle to the left of ℓ_2 with its base spanning from m_2 to m_0, and split it into an isosceles triangle congruent to $B_{k/2}$ sitting atop an isosceles trapezoid B_1. Cut a second isosceles trapezoid B_1 that shares its base with the first isosceles trapezoid, and is thus congruent to the first.
6. This leaves $k/2$ pairs of congruent pieces to the left of ℓ_k, with pieces C_i in the slices on either side of ℓ_{2i-1}.

There are $2k$ pieces in each of the two sets of slices, namely the k mirror imaged pieces A_1 through A_k (from step 3), the $k/2$ pieces that possess bilateral symmetry, namely one isosceles trapezoid B_1 (from step 5) and $k/2 - 1$ isosceles triangles B_2 through $B_{k/2}$ (from steps 4 and 5), and lastly, the $k/2$ congruent pieces C_1 through $C_{k/2}$ (from step 6). In mapping from the white to the shaded pieces, k pieces A_1 through A_k are turned over, and k pieces B_1 through $B_{k/2}$ and C_1 through $C_{k/2}$ are not turned over.

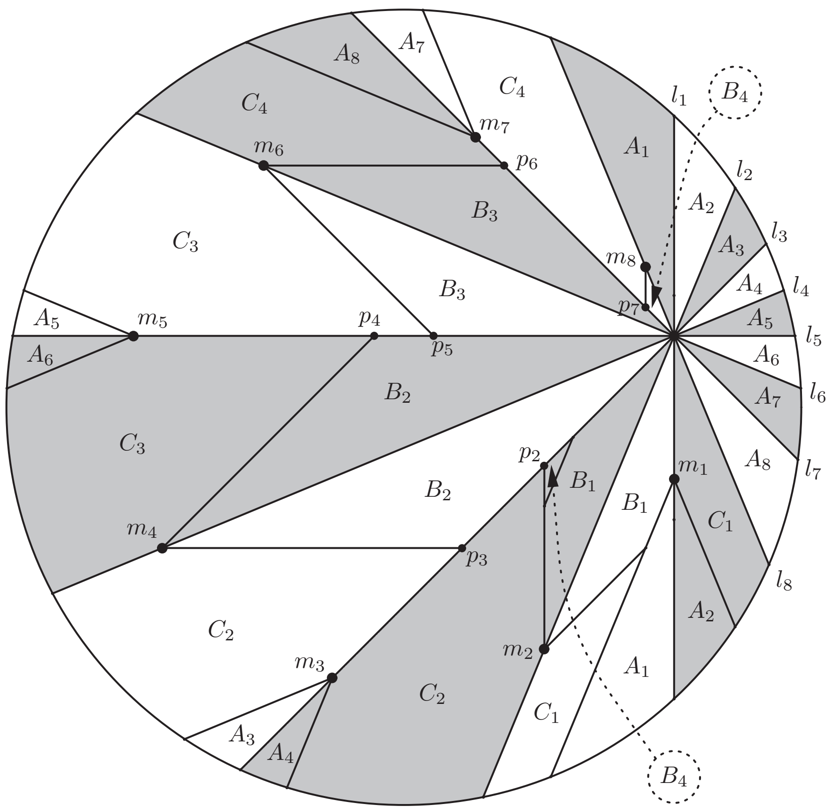


Figure 4 Our dissection for 8 chords, with the ℓ_i, m_i, and p_i labeled

The dissection for the case of 8 chords is shown in FIGURE 4, with the ℓ_i and m_i labeled. For $i = 3, 5, \ldots, k - 1$, we name the apexes of the isosceles triangles so that apexes p_{i-1} and p_i fall on chord ℓ_i, as shown in FIGURE 4. Given our examples for 4, 6, and 8 chords, producing the diagram for the dissection of the disk with 10 chords should be a piece of cake—er, pizza!

Ensuring that our eyes are no bigger than our stomachs

We can interpret our diagrams as "visual proofs," but that can be dangerous, because things may look okay even when they are not. How can we make sure that our eyes do not lead us astray? We need to verify that our procedure produces a correct dissection for any even $k \geq 4$. To achieve this, we argue several things: that we can generate a labeling of chords as in step 2, that the mirror images of A_{2i-1} and A_{2i} land in larger slices that are "opposite," that the pieces we call "congruent pieces" really are congruent, and that the apex of each triangle B_i falls within the disk. Choose the top slice to the left of ℓ_1 to be shaded, which forces each piece to be white or shaded.

First, we can label our chords in step 2 appropriately, because we establish the property that all m_i fall on a certain circle:

PROPERTY $\mathcal{C}$ (CIRCLE). *Let m_{-1} be the unlabeled vertex of white trapezoid B_1. Then all of the m_i for $i = -1, 0, \ldots, k$, fall on a circle whose center coincides with the center of the disk.*

In proof, note that the perpendicular bisector of each edge m_0m_i, $i = 1, 2, \ldots, k$, goes through the center of the disk. Since the perpendicular bisectors of all three sides of a triangle $m_0m_im_{i-1}$ meet at the same point, then the perpendicular bisector of edge m_im_{i-1} also goes through the center of the disk. Furthermore, the perpendicular bisectors of edges m_0m_2 and $m_{-1}m_1$ are identical, so the perpendicular bisector of m_2m_{-1} also goes through the center of the disk. Then the desired circle is the circumcircle of polygon $m_0m_1m_{-1}m_2m_3 \cdots m_k$.

Property $\mathcal{C}$ also guarantees that the mirror images of A_{2i-1} and A_{2i} land appropriately, because the mirror image of m_0 on ℓ_{2i-1} lands on point m_{2i-1} of the circle. The slices opposite A_{2i-1} or A_{2i} have greater area and thus more curved boundary.

Next, we show that the pieces we designate as "congruent pieces" really are congruent. When m_0 is not on the boundary of the disk, all C_i except for C_1 have four straight sides and one curved side. Let's first consider any piece C_i with $i > 1$. It is easy to see that the respective angles of white piece C_i are equal to those of shaded piece C_i. Name the sides of white piece C_i clockwise as w_0, w_1, w_2, w_3, and w_4, with w_0 being the curved side. Similarly, name the sides of shaded piece C_i clockwise as s_0, s_1, s_2, s_3, and s_4, with s_0 being curved.

Since w_1 is the mirror image of the edge separating white piece A_{2i} and shaded piece A_{2i+1} across the diameter that is perpendicular to ℓ_{2i} (which separates A_{2i} and A_{2i+1}), those two edges have the same length. Since s_1 is the mirror image of the edge separating white piece A_{2i} and shaded piece A_{2i+1} across the diameter that is perpendicular to ℓ_{2i-1} (which separates white piece A_{2i-1} and shaded piece A_{2i}), those two edges have the same length. Thus w_1 and s_1 have equal length. Similarly, so do w_4 and s_4.

To handle curved sides w_0 and s_0, we recall a theorem about the angles of intercepted chords, namely that the measure of the angle formed by two chords that intersect within a circle is one half the sum of the chords' intercepted arcs. Chords ℓ_{2i-1} and ℓ_{2i} which sandwich white piece C_i also sandwich white pieces A_{2i-1} and A_{2i}, and these are the only pieces between those two chords that contain boundary arcs. The analogous condition holds for chords ℓ_{2i-2} and ℓ_{2i-1} and shaded pieces C_i, A_{2i-1}, and A_{2i}. Since pieces A_{2i-1} are mirror images of each other, as are pieces A_{2i}, their arc lengths are equal. Since the angle between ℓ_{2i-1} and ℓ_{2i} equals the angle between ℓ_{2i-2} and ℓ_{2i-1}, it follows from the theorem that w_0 and s_0 have equal length.

Now we imagine white piece C_i split into a 4-sided piece, with sides w_0, w_1, w_4, and a new side w_5, and a 3-sided piece with sides w_2, w_3, and the new side w_5, and do the same for shaded piece C_i, naming the new side s_5. Then the two 4-sided pieces are congruent, since they match on three consecutive sides and the two intervening angles.

Thus w_5 and s_5 have the same length and their adjacent angles also match. In the 3-sided figures, w_5 and s_5 have the same length, and their adjacent angles match. Thus the 3-sided figures are also congruent. It follows that the white piece C_i is congruent to the shaded piece C_i.

It remains to show that the pair of pieces with three straight sides and one curved side, namely pieces C_1, are indeed congruent. Again note that the respective angles of white piece C_1 are equal to those of shaded piece C_1. Then name the sides clockwise in the white piece C_1 as w_0, w_1, w_2, and w_3, with w_0 being the curved side. Similarly, name the sides clockwise in the shaded piece C_1 as s_0, s_1, s_2, and s_3, with s_0 being the curved side.

As with piece C_i for $i > 1$, we argue that w_1 and s_1 for C_1 are the same length, and similarly for w_3 and s_3. As for w_2, it is of length equal to the length of one of the equal sides of isosceles trapezoid B_1, which is the same as the length of s_2. Thus pieces C_1 are congruent, and all pieces claimed to be congruent are indeed congruent.

In the case that m_0 lies on the boundary of the disk, each C_i has fewer sides. If we instead view each C_i as having the same number of sides, but with some of them of length 0, then our previous arguments apply.

Next, we establish a pretty mirror-image property relating to the apexes of the isosceles triangles:

PROPERTY $\mathcal{M}$ (MIRROR). *For $i = 3, 5, \dots, k-1$, apexes p_{i-1} and p_i on chord ℓ_i are mirror images of each other across the midpoint of ℓ_i.*

To prove Property $\mathcal{M}$, we focus on the pieces $C_{\lceil i/2 \rceil}$. Since white piece $C_{\lceil i/2 \rceil}$ is congruent to shaded piece $C_{\lceil i/2 \rceil}$, the length $\overline{m_{i+1}p_i}$ equals the length $\overline{m_i p_{i-1}}$. Since white $B_{\lceil i/2 \rceil}$ is an isosceles triangle, the length $\overline{m_{i+1}p_i}$ equals the length $\overline{m_0 p_i}$. Then the length $\overline{m_i p_{i-1}}$ equals the length $\overline{m_0 p_i}$. Since m_i and m_0 are mirror images of each other across the midpoint of ℓ_i, it follows that p_{i-1} and p_i are mirror images of each other across the midpoint of ℓ_i.

Finally, we establish that each apex p_i, for $i = 2, 3, \dots, k-1$, falls inside the disk. Point p_{k-1} is between the midpoint of edge $m_k m_0$ and the center of the disk, and thus inside the disk. By $\mathcal{M}$, p_{k-2} is also between m_0 and m_{k-1} on ℓ_{k-1}. Let j be the largest index such that p_j is below the horizontal diameter of the disk. Then for each $i > j$, where p_i is an apex of a shaded isosceles triangle, p_i and p_{i-1} are on the perpendicular bisector of ℓ_i, and on the same side of the bisector from the center of the disk. Thus if p_i is inside the disk, then by $\mathcal{M}$ so is p_{i-1}.

For $i \le j$, we work from the other direction: For $j > 2$, p_2 is between the midpoint of edge $m_2 m_0$ and the center of the disk, and thus inside the disk. By $\mathcal{M}$, p_3 is also inside the disk. Then for each $i < j-1$, where p_i is the apex of a white isosceles triangle, p_i and p_{i-1} are on the perpendicular bisector of ℓ_{i+1} and on the same side of the bisector from the center of the disk. And if p_{i-1} is inside the disk, then by $\mathcal{M}$ so is p_i. Thus all of $p_2, p_3, \dots, p_{k-1}$ are inside the disk.

Note that by Property $\mathcal{C}$ the two apexes m_1 and m_{-1} of white trapezoid B_1 fall inside the disk. Thus the apexes of all isosceles pieces fall inside the disk, and our dissection is correct.

Food for further thought

There are additional specialty pizza results to enjoy. For a pizza sliced as in FIGURE 1, Rick Mabry and Paul Deiermann showed that two people will get an equal amount of crust whenever they get an equal amount of pizza [**10**], whether the pizza has a "thin crust," represented by its boundary, or a "thick crust," represented by an annulus. Our results give an alternate, and no less elegant, proof of this fact for the thin-crust case: Our dissection of one set of slices gives pieces that we either move by translation, or flip over and then move, to give the other set of slices. Each curved boundary in our pieces originates from the boundary of the disk, so each curved boundary in our pieces is either moved, or flipped and moved. So the amount of thin crust in each of the two sets of slices is identical.

There is an even tastier result that we can produce with our dissections. Assuming that the chords of FIGURE 2 are a subset of the chords of FIGURE 4, if we match up pairs of adjacent slices in FIGURE 4 appropriately with the slices in FIGURE 2, we can see that matching pieces in FIGURE 4 land in different sets of slices, except for pieces B_1, B_2, and B_3. In fact pieces B_1, B_1, B_3, and B_3 land in one set of the slices, and pieces B_2 and B_2 land in the other set of slices. Since the total area of pieces in one set of slices equals the total area in the other, we conclude that the total area of pieces in set $S_1 = \{B_1, B_3\}$ equals the area of the piece in set $S_2 = \{B_2\}$.

We can also match up pairs of adjacent slices in FIGURE 4 in a different way, so that the pieces landing in different sets of slices are just pieces B_1, B_2, and B_3. Then pieces A_1, A_1, A_2, A_2, A_5, A_5, A_6, A_6, C_1, C_1, C_3, and C_3 land in one set of slices, and A_3, A_3, A_4, A_4, A_7, A_7, A_8, A_8, C_2, C_2, C_4, C_4, B_4, and B_4 land in the other set of slices. We conclude that the total area of pieces in $S_3 = \{A_1, A_2, A_5, A_6, C_1, C_3\}$ equals the total area of pieces in $S_4 = \{A_3, A_4, A_7, A_8, C_2, C_4, B_4\}$. Then we get four different sets of pieces, $S_1 \cup S_3$, $S_1 \cup S_4$, $S_2 \cup S_3$, and $S_2 \cup S_4$, which correspond to what we get if we take every fourth slice from FIGURE 4. Thus we get four equal shares from a pizza divided by 8 chords. This is a dissections-based proof of part of what the Hirschhorns proved analytically in [**7**].

Finally let's explore similar results in three dimensions, namely dissecting a (spherical) ball. Identify as the *primary plane section* some plane section of the ball. Also identify a line segment that is the intersection of the ball with a line that is perpendicular to the primary plane section at a point inside the ball. For any even number $k \geq 4$, identify k plane sections that contain the line segment and are equally spaced around it. The $k + 1$ plane sections partition the ball into $4k$ segments. Suppose that none of the plane sections contains the center of the ball. Color $2k$ of the segments white and the other $2k$ shaded, so that any white segment shares boundaries of positive area only with shaded segments, and vice versa. Then the "calzone theorem" states that the total volume of white segments equals the total volume of shaded segments. In 1994, George Berzsenyi announced that Michael Nathanson, then an undergraduate at Brown University, had found an analytic proof, and Berzsenyi challenged readers to find a dissection proof [**1**].

A dissection proof of the calzone theorem is an easy corollary of our 2-dimensional proof: Identify the intersection point of the $k + 1$ plane sections, and identify its mirror image point on the line segment. Let the *mirror plane section* be the plane section that is parallel to the primary plane section and contains the mirror image point. FIGURE 5 shows $k + 1$ plane sections and the mirror plane section for the case where $k = 4$. Cut segments with the mirror plane section, naming the pieces between the primary plane section and the mirror plane section *slab pieces*. Cut the slab pieces in a fashion similar to our dissection of the disk in the pizza problem. Since each non-slab piece U_i can be paired with a mirror image piece D_i of the opposite color, we have a dissection into $4k$ white pieces and $4k$ shaded pieces, such that each of the $2k$ white slab pieces is congruent to a shaded slab piece, and each of the $2k$ white non-slab pieces is the mirror image of a shaded non-slab piece. The calzone theorem then follows.

In conclusion, we have given a general dissection method that establishes equality of area for the two sets of slices resulting from k equally-spaced chordal cuts of a disk. Not only does our method also establish the equality of the total length of the curved

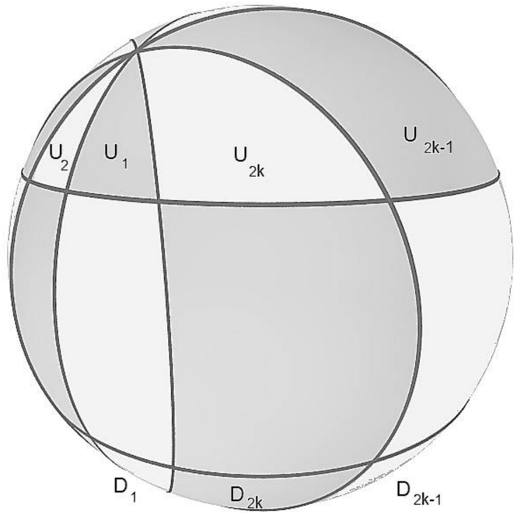


Figure 5 A calzone with the $k+1$ plane sections and the mirror plane section

boundaries in each set, but it can also be used to establish equality of area for four sets of slices as well. And, as in any good math problem, our methods extend even further. So, all in all, who says that mathematicians can't have their pizza and eat it too?

Note added in proof. After this paper was accepted for publication, Rick Mabry recalled from [**1**] that Allen Schwenk had found dissection proofs for both $k = 6$ and $k = 8$ chords. These proofs were never published, and Allen did not remember producing a dissection for $k = 8$. He eventually located a number of diagrams, including one that is essentially the same as Figure 4.

Acknowledgment Thanks to Stan Rabinowitz for information about L. J. Upton, to Allen Schwenk for permission to include his dissection for 6 chords, and to Rick Mabry, Stan Wagon, and the referees for many helpful suggestions.

REFERENCES

1. George Berzsenyi, The pizza theorem—part II, *Quantum* **4**(4) (1994) 1994.
2. Larry Carter and Stan Wagon, Proof without words: Fair allocation of a pizza, *Mathematics Magazine* **67**(4) (1994) 267. http://dx.doi.org/10.2307/2690845.
3. Greg N. Frederickson, *Dissections Plane & Fancy*, Cambridge University Press, New York, 1997.
4. Greg N. Frederickson, Casting light on cube dissections, *Mathematics Magazine* **82**(5) (2009) 323–331.
5. Greg N. Frederickson, Polishing some visual gems, *Math Horizons* (September 2009) 21–25. http://dx.doi.org/10.4169/194762109X468300.
6. Michael Goldberg, Divisors of a circle: solution to problem 660, *Mathematics Magazine* **41**(1) (1968) 46.
7. Jeremy Hirschhorn, Michael D. Hirschhorn, Jeremy K. Hirschhorn, Andrew D. Hirschhorn, and Philip M. Hirschhorn, The pizza theorem, *Gazette of the Australian Mathematical Society* **26**(3) (1999) 120–121.
8. Joseph D. E. Konhauser, Dan Velleman, and Stan Wagon, *Which Way Did the Bicycle Go?* Dolciani Mathematical Expositions, No. 18, Mathematical Association of America, Washington, DC, 1996.
9. Harry Lindgren, *Geometric Dissections*, Van Nostrand, Princeton, NJ, 1964.
10. Rick Mabry and Paul Deiermann, Of cheese and crust: A proof of the pizza conjecture and other tasty results, *American Mathematical Monthly* **116**(5) (2009) 423–438. http://dx.doi.org/10.4169/193009709X470317.
11. Stanley Rabinowitz, Ptolemy's legacy, book-length manuscript.
12. L. J. Upton, Problem 660, *Mathematics Magazine* **40**(3) (1967) 163.

Summary Symmetry, circumcircles, and angles of intercepted chords are some of the ingredients, as mathematicians give dissection proofs of the pizza theorem to show that they can get their fair share, even when the original pizza cuts are off-center. And with just a slight variation in the recipe, we also get a dissection proof of the calzone theorem.

A Galois Connection in the Social Network

JAMES PROPP
Department of Mathematical Sciences
University of Massachusetts, Lowell, MA 01854
http://jamespropp.org

Assume that knowing is a symmetric relation, so that x knows y if and only if y knows x. (This symmetry holds for some sorts of acquaintanceship, such as the "friending" relationship on Facebook.)

THEOREM.

The people who know
all the people who know
all the people you know
all are people you know

and the people you know
all are people who know
all the people who know
all the people you know.

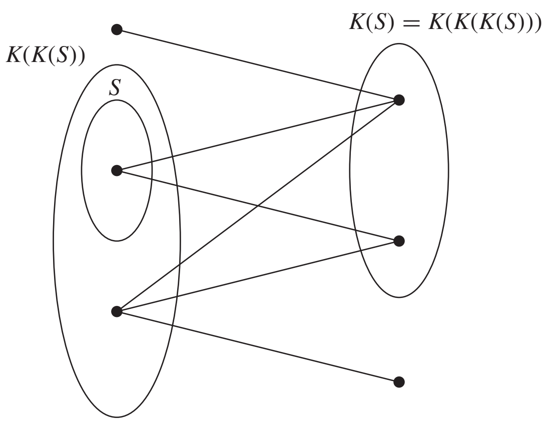


Figure 1 A small social network

Proof. For any set S of people, let $K(S)$ denote the set of people who know *everyone* in S. The accompanying figure shows a small bipartite example, with dots representing people and with edges joining people who know each other. K is *inclusion-reversing:* $S \subseteq S'$ implies $K(S') \subseteq K(S)$.

It is not hard to see that

$$S \subseteq K(K(S)) \tag{1}$$

Math. Mag. **85** (2012) 34–36. doi:10.4169/math.mag.85.1.34. © Mathematical Association of America

Applying the inclusion-reversing property to (1) yields

$$K(K(K(S))) \subseteq K(S) \tag{2}$$

On the other hand, replacing S by $K(S)$ in (1) gives

$$K(S) \subseteq K(K(K(S))) \tag{3}$$

The two stanzas of the theorem are obtained by specializing (2) and (3) to the case $S = \{\text{you}\}$. ∎

The mathematical claim and its proof are not original. The operation $K(\cdot)$ is an example of an antitone *Galois connection* from the power set of U to itself, where U is the universe of people. The general notion of a Galois connection can be traced at least as far back as Birkhoff [**1**]; see also the other listed references. An antitone Galois connection is a pair of functions $F : A \to B$ and $G : B \to A$ between two partially ordered sets A and B, such that for all a in A and b in B, $b \leq_B F(a)$ if and only if $a \leq_A G(b)$. In our case, A and B are both the power set of the universe of people, ordered by inclusion, and F and G are both the map K. To see that we have a Galois connection, note that "$b \leq_B F(a)$" is tantamount to the proposition "everyone in the set b knows everyone in the set a", while "$a \leq_A G(b)$" is tantamount to the equivalent proposition "everyone in the set a knows everyone in the set b".

The pair of relations *likes* and *is liked by* also gives rise to a Galois connection, where $F(a)$ is the set of people whom everyone in a likes, and $G(b)$ is the set of people who like everyone in b. The proof of the theorem given above is a specialization of the proof that for any Galois connection, $F \circ G \circ F = F$ and $G \circ F \circ G = G$.

Many Galois connections occur in bipartite settings, where the sets A and B are disjoint, and indeed the term originates from one such example that long predates Birkhoff: given a Galois extension E of a number field F, the pair of relations *likes* and *is liked by* treated above can be replaced by the pair of relations *fixes* and *is fixed by*, where a group-element σ of $\mathrm{Gal}(E/F)$ fixes a field-element x of E if and only if $\sigma(x) = x$. The sets A and B respectively consist of the subfields of E/F and the subgroups of $\mathrm{Gal}(E/F)$, ordered by inclusion. As part of the proof of the Fundamental Theorem of Galois Theory, one shows that, for any subfield K of E/F, the group-elements that fix all the field-elements that are fixed by all the group-elements that fix K all are group-elements that fix K, and vice versa; these are precisely the automorphisms of E/K. Furthermore, the field-elements that are fixed by all the group-elements that fix K themselves form a field, namely the *Galois closure* of K, which contains K. In the social network, one has an analogous closure operator sending S to the set $K(K(S)) \supseteq S$.

Also, if a topological space X is a path-connected, there is a Galois connection between subgroups of the fundamental group of X and path-connected covering spaces of X. The book [**4**] shows how this idea from algebraic topology can be applied to the study of Fuchsian differential equations.

A consequence of (2) and (3) is the equality $K(K(K(S))) = K(S)$. One virtue of stating the proof in two stanzas—that is, expressing it as mutual inclusion of sets rather than equality between sets—is that it gives a hint of the proof. As a bonus, the theorem as worded can be sung fluidly (albeit incomprehensibly) to the tune of the jig "The Irish Washerwoman" (`http://www.ireland-information.com/irishmusic/theirishwasherwoman.shtml`).

Acknowledgments Thanks to Henry Crapo, David Feldman and Neil Strickland for identifying relevant references.

REFERENCES

1. Garrett Birkhoff, Lattice theory, *Amer. Math. Soc. Coll. Pub.* **25** (1940).
2. Henry Crapo, Ordered sets: Retracts and connections, *J. Pure and Appl. Alg.* **23** (1982) 13–28. http://dx.doi.org/10.1016/0022-4049(82)90074-3.
3. M. Erne, J. Koslowski, A. Melton, and G. E. Strecker, A primer on Galois connections, *Ann. New York Acad. Sci.* **704** (1993) 103–125. http://dx.doi.org/10.1111/j.1749-6632.1993.tb52513.x; also at http://www.math.ksu.edu/~strecker/primer.ps.
4. Michio Kuga, *Galois' Dream: Group Theory and Differential Equations*, Birkhauser, 1992.
5. Øystein Ore, Galois connexions, *Trans. Amer. Math. Soc.* **55** (1944) 493–513.
6. Peter Smith, The Galois connection between syntax and semantics; available at http://www.logicmatters.net/resources/pdfs/Galois.pdf.
7. Wikipedia article http://en.wikipedia.org/wiki/Galois_connection.

Summary This article shows that if "knowing" is a symmetric relation, then the set of people who know all of the people who know all of the people who know everyone in S coincides with the set of people who know everyone in S. In symbols this becomes $K(K(K(S))) = K(S)$. This fact fits into the broader context of Galois connections between partially ordered sets.

First Digits of Squares and Cubes

KENNETH A. ROSS
University of Oregon
Eugene, OR 97403
rossmath@pacinfo.com

Benford's Law is the observation that in many lists of numbers that arise in the real world, the first digit is 1 about 30% of the time, it is 2 over 17% of the time, and so on with the first digit being 9 only about 5% of the time. More specifically, for $d = 1, 2, \ldots, 9$, the likelihood that the first digit is d is $\log_{10}(d+1) - \log_{10} d = \log_{10}\left(\frac{d+1}{d}\right)$. Approximations of these values are in TABLE 1.

TABLE 1:

$d \longrightarrow$	1	2	3	4	5	6	7	8	9
$\log_{10}\frac{d+1}{d}$	0.3010	0.1761	0.1249	0.0969	0.0792	0.0669	0.0580	0.0511	0.0458

Though this first-digit phenomenon was observed as far back as 1881, there's been a recent surge of interest, especially in the past 15 years. The website Benford Online Bibliography [**2**] has over 600 entries regarding Benford's law, mostly articles but some website and video items as well.

Research articles that explain the amazing generality of Benford's Law rely on sophisticated ideas from probability, but in [**8**] I provide a simple explanation, suitable for nonmathematicians, for the case when the observed data have been growing (or shrinking) exponentially over time. (No simple explanation is known for the general phenomenon; see [**1**].) Examples of exponential growth include populations of counties or cities and the values of portfolios. In fact, I start with a concrete non-random example, by looking at powers of 2 (2, 4, 8, 16, ...), and observing that their first digits exhibit a similar Benford-like phenomenon. That is, the *long-run relative frequency*

Math. Mag. **85** (2012) 36–42. doi:10.4169/math.mag.85.1.36.

of each digit d is $\log_{10}(d+1) - \log_{10} d$. If for any $x > 0$ we write $D(x)$ for the first digit of x, then we can state this as follows:

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{k : 1 \le k \le n \quad \text{and} \quad D(2^k) = d\right\} = \log_{10}(d+1) - \log_{10} d, \qquad (1)$$

for $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$. (Here and elsewhere, k, n and m always represent integers.) Assertion (1) is a special case of Theorem 1 in [**8**], which tells us that if r is a positive rational number that is not an integer power of 10, and if a is any positive number, then

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{k : 1 \le k \le n \quad \text{and} \quad D(ar^k) = d\right\} = \log_{10}(d+1) - \log_{10} d, \qquad (2)$$

for $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$. Assertion (2) is a direct consequence of the 100-year-old Bohl-Sierpiński-Weyl Uniform Distribution Theorem; for a statement of that theorem and further references, see [**8**, pp. 576–578].

It is natural to ask whether a similar result holds for the sequences $\{k^2\}$ and $\{k^3\}$. In fact, (2) fails for every sequence $\{k^M\}$ where M is a fixed positive integer; this follows from [**4**, Corollary 2]. We'll prove a slightly stronger statement:

THEOREM 1. *For integers $M \ge 2$ and $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$, the limits*

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{k : 1 \le k \le n \quad \textit{and} \quad D(k^M) = d\right\}$$

do not exist.

Proof. First we illustrate with $M = d = 2$. Assume that the limit L exists; because of Theorems 2 and 3 below, we must have $L < 1$. Observe that for all n,

$$1 = \frac{1}{2 \cdot 10^n} \#\{k : 15 \cdot 10^n + 1 \le k \le 17 \cdot 10^n \quad \text{and} \quad D(k^2) = 2\},$$

and that this number equals

$$\begin{aligned} &8.5 \cdot \frac{1}{17 \cdot 10^n} \#\{k : 1 \le k \le 17 \cdot 10^n \quad \text{and} \quad D(k^2) = 2\} \\ &-7.5 \cdot \frac{1}{15 \cdot 10^n} \#\{k : 1 \le k \le 15 \cdot 10^n \quad \text{and} \quad D(k^2) = 2\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $1 = 8.5L - 7.5L = L < 1$, a contradiction.

Now consider general M and d, and assume that the limit L exists, $L < 1$. Select finite decimals A and B satisfying $\sqrt[M]{d} < A < B < \sqrt[M]{d+1}$. Then the M-th powers of all numbers between A and B have first nonzero digit d, and for large n, $10^n A$ and $10^n B$ will be integers. So for large n,

$$1 = \frac{1}{10^n(B-A)} \#\{k : 10^n A + 1 \le k \le 10^n B \quad \text{and} \quad D(k^M) = d\},$$

and this equals

$$\begin{aligned} &\frac{B}{B-A} \cdot \frac{1}{10^n B} \#\{k : 1 \le k \le 10^n B \quad \text{and} \quad D(k^M) = d\} \\ &-\frac{A}{B-A} \cdot \frac{1}{10^n A} \#\{k : 1 \le k \le 10^n A \quad \text{and} \quad D(k^M) = d\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$1 = \frac{B}{B-A}L - \frac{A}{B-A}L = L < 1,$$

a contradiction. ■

In spite of Theorem 1, we will see that interesting subsequences do have limits. For example,

$$\lim_{n\to\infty} \frac{1}{10^n - 1} \# \left\{ k : 1 \le k < 10^n \quad \text{and} \quad D(k^2) = d \right\} = \frac{\sqrt{d+1} - \sqrt{d}}{\sqrt{10} - 1}, \tag{3}$$

and more generally,

THEOREM 2. *For integers $M \ge 2$ and $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$,*

$$\lim_{n\to\infty} \frac{1}{10^n - 1} \# \left\{ k : 1 \le k < 10^n \quad \textit{and} \quad D(k^M) = d \right\} = \frac{\sqrt[M]{d+1} - \sqrt[M]{d}}{\sqrt[M]{10} - 1}. \tag{4}$$

Approximate values of the limits in (3) and (4) are given in TABLE 2, where TABLE 1 is repeated for comparison.

TABLE 2:

$d \longrightarrow$	1	2	3	4	5	6	7	8	9
$M = 2$	0.1916	0.1470	0.1239	0.1092	0.0987	0.0908	0.0845	0.0793	0.0750
$M = 3$	0.2252	0.1579	0.1257	0.1062	0.0928	0.0830	0.0754	0.0694	0.0644
$M = 10$	0.2772	0.1713	0.1258	0.1001	0.0835	0.0718	0.0631	0.0563	0.0510
$M = 100$	0.2986	0.1756	0.1250	0.0972	0.796	0.0674	0.0585	0.0517	0.0463
$\log_{10} \frac{d+1}{d}$	0.3010	0.1761	0.1249	0.0969	0.0792	0.0669	0.0580	0.0511	0.0458

As suggested by TABLE 2, using L'Hospital's rule, we have

$$\lim_{M\to\infty} \frac{\sqrt[M]{d+1} - \sqrt[M]{d}}{\sqrt[M]{10} - 1} = \log_{10}(d+1) - \log_{10} d, \tag{5}$$

the very values that arise as limits in Benford's Law.

TABLE 3 gives the actual count of numbers k^2 (where k runs from 1 to 10^n) that begin with each digit $d = 1, 2, \ldots, 9$. The table suggests that the sequences in (3) converge very rapidly.

TABLE 3:

$d \longrightarrow$	1	2	3	4	5	6	7	8	9
100	21	14	12	12	9	9	8	7	8
1000	194	146	123	111	97	91	84	78	76
10000	1919	1469	1237	1095	984	908	845	791	752
limit in (3)	0.1916	0.1470	0.1239	0.1092	0.0987	0.0908	0.0845	0.0793	0.0750

Even though the limit in Theorem 1 does not exist, from its proof it seemed plausible that Theorem 2 might hold, because as k runs from 10^j to 10^{j+1}, the first digits of $\{k^M\}$ run through complete cycles $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$, so that each digit has a "fair" chance of appearing.* In fact, the first digits of $\{k^M\}$ run through M complete cycles as k runs from 10^j to 10^{j+1}. Specifically, the first digits of $\{k^M\}$ run through one complete cycle as k runs through each interval $[10^{m/M}, 10^{(m+1)/M})$. This suggests that the following stronger theorem might be true:

THEOREM 3. *For integers $M \geq 2$ and $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$,*

$$\lim_{n\to\infty} \frac{1}{\lceil 10^{n/M}\rceil - 1} \#\left\{k : 1 \leq k < 10^{n/M} \quad \text{and} \quad D(k^M) = d\right\} = \frac{\sqrt[M]{d+1} - \sqrt[M]{d}}{\sqrt[M]{10} - 1}.$$

Here $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. This interesting theorem was obtained in 2004 by Werner Hürlimann [**6**]. Note that Theorem 2 follows from Theorem 3, because the sequences in Theorem 2 are subsequences of the corresponding sequences in Theorem 3. We prove the following generalization of Theorem 3.

THEOREM 4. *For any real number $c > 0$ and $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$,*

$$\lim_{n\to\infty} \frac{1}{\lceil 10^{n/c}\rceil - 1} \#\left\{k : 1 \leq k < 10^{n/c} \quad \text{and} \quad D(k^c) = d\right\} = \frac{(d+1)^{1/c} - d^{1/c}}{10^{1/c} - 1}. \quad (6)$$

TABLE 4 illustrates Theorem 4 for $c = 3/4$. It gives the count of numbers $k^{3/4}$ (where k runs from 1 to $\lceil 10^{4n/3}\rceil$, instead of from 1 to $\lceil 10^{4n/3}\rceil - 1$) that begin with each digit $d = 1, 2, \ldots, 9$.

TABLE 4:

$d \longrightarrow$	1	2	3	4	5	6	7	8	9
465 ($n = 2$)	36	41	45	50	52	57	58	62	64
percents of 465	0.0774	0.0882	0.0968	0.1075	0.1118	0.1226	0.1247	0.1333	0.1376
10000 ($n = 3$)	741	880	984	1071	1144	1212	1269	1325	1374
limit in (6)	0.0740	0.0880	0.0985	0.1071	0.1145	0.1211	0.1270	0.1324	0.1374

For large values of m, the intervals $[10^{m/c}, 10^{(m+1)/c})$ will play a key role in the proof of Theorem 4. Since $\lim_{m\to\infty}[10^{(m+1)/c} - 10^{m/c}] = \infty$, these intervals get longer and longer. To avoid vacuous and trivial cases, we select an integer m_c so that these intervals have at least two integers for $m \geq m_c$. For $m \geq 0$, we write $A_m = \lceil 10^{m/c}\rceil$. Then for $m \geq m_c$,

$$A_m \text{ and } A_{m+1} - 1 \text{ are the smallest and largest integers in } [10^{m/c}, 10^{(m+1)/c}), \quad (7)$$

respectively.

Also, we will use in the proof what I call "almost Riemann sums," i.e., Riemann sums where one or both of the end intervals of the partitions need not be used. Given a (bounded) Riemann integrable function on a bounded interval, it is a simple exercise to prove that "almost Riemann sums," whose meshes go to 0, also converge to the appropriate integrals, because they differ from the corresponding real Riemann sums by at most the values using the end intervals of the partitions.

*This is clear for small M, like 2 and 3, but for large M, some complete cycles described here will be empty. In the proof of Theorem 4, we will sidestep this issue.

Proof of Theorem 4. For $n \geq m_c$, let

$$t_n = \frac{1}{A_n - 1} \# \left\{ k : 1 \leq k < 10^{n/c} \quad \text{and} \quad D(k^c) = d \right\};$$

throughout this proof, c and d are fixed. We need to show

$$\lim_{n \to \infty} t_n = \frac{(d+1)^{1/c} - d^{1/c}}{10^{1/c} - 1}. \tag{8}$$

For $m \geq m_c$, let

$$s_m = \frac{1}{A_{m+1} - A_m} \# \{ k : A_m \leq k \leq A_{m+1} - 1 \quad \text{and} \quad D(k^c) = d \},$$

let

$$a_{m,n} = \frac{A_{m+1} - A_m}{A_n - 1} \quad \text{for} \quad m_c \leq m \leq n - 1,$$

and let $a_{m,n} = 0$ for $m_c \leq n \leq m$. Then for $n \geq m_c$, we have

$$t_n = \frac{1}{A_n - 1} \sum_{m=m_c}^{n-1} (A_{m+1} - A_m) s_m = \sum_{m=m_c}^{\infty} a_{m,n} s_m.$$

Now $(a_{m,n})_{m,n=m_c}^{\infty}$ is an infinite matrix satisfying

$$a_{m,n} \geq 0 \text{ for all } m, n, \ \sum_{m=m_c}^{n-1} a_{m,n} = 1 \text{ for each } n,$$

and $\lim_{n \to \infty} a_{m,n} = 0$ for each $m \geq m_c$; compare [**10**, Theorem 7.85], which is due to Toeplitz. A short ϵ-m-n argument, using the identity $t_n - L = \sum_{m=m_c}^{n-1} a_{m,n}(s_m - L)$, shows that

$$\lim_{m \to \infty} s_m = L \quad \text{implies} \quad \lim_{n \to \infty} t_n = L.$$

Thus, to verify (8), it suffices to prove

$$\lim_{m \to \infty} s_m = \frac{(d+1)^{1/c} - d^{1/c}}{10^{1/c} - 1}, \tag{9}$$

for $d = 1, 2, 3, 4, 5, 6, 7, 8, 9$. If we divide the numbers in

$$\{ k : A_m \leq k \leq A_{m+1} - 1 \quad \text{and} \quad D(k^c) = d \}$$

by $10^{m/c}$, and notice that $D((k/10^{m/c})^c) = D(k^c)$ for all k, we see that

$$s_m = \frac{1}{A_{m+1} - A_m} \# \left\{ \frac{k}{10^{m/c}} : \frac{A_m}{10^{m/c}} \leq \frac{k}{10^{m/c}} \leq \frac{A_{m+1} - 1}{10^{m/c}} \right.$$
$$\left. \quad \text{and} \quad D((k/10^{m/c})^c) = d \right\},$$

for $m \geq m_c$. Using (7), we see that the numbers $\frac{k}{10^{m/c}}$, for $A_m \leq k \leq A_{m+1} - 1$, plus 1 on the left and $10^{1/c}$ on the right, form a partition of $[1, 10^{1/c}]$ having mesh $\frac{1}{10^{m/c}}$. For x

in $[1, 10^{1/c})$, we have $D(x^c) = d$ if and only if x is in $[d^{1/c}, (d+1)^{1/c})$. Because of this we define a function f_d on $[1, 10^{1/c})$ so that it takes the value 1 on $[d^{1/c}, (d+1)^{1/c})$ and 0 elsewhere. It follows that

$$s_m = \frac{1}{A_{m+1} - A_m} \sum_{k=A_m}^{A_{m+1}-1} f_d\left(\frac{k}{10^{m/c}}\right).$$

Now each

$$r_m = \frac{1}{10^{m/c}} \sum_{k=A_m}^{A_{m+1}-1} f_d\left(\frac{k}{10^{m/c}}\right)$$

is an "almost Riemann sum" for f_d on $[1, 10^{1/c}]$, and we already noted that the meshes of their partitions go to 0, as $m \to \infty$. Therefore

$$\lim_{m\to\infty} r_m = \int_1^{10^{1/c}} f_d(x)\,dx = (d+1)^{1/c} - d^{1/c}.$$

Since

$$s_m = \frac{10^{m/c}}{A_{m+1} - A_m} \cdot r_m \quad \text{and} \quad \lim_{m\to\infty} \frac{10^{m/c}}{A_{m+1} - A_m} = \frac{1}{10^{1/c} - 1},$$

we conclude that

$$\lim_{m\to\infty} s_m = \frac{(d+1)^{1/c} - d^{1/c}}{10^{1/c} - 1}.$$

This establishes (9), hence (8) and hence (6), i.e., Theorem 4. ■

Note. Where Benford's Law holds, the likelihoods of the digits in different positions also have been studied. As with first digits, they are not uniformly distributed. See [**5**] for details. It is especially interesting that the probabilities are dependent. For example, the (unconditional) probability that the second digit is 2 is ≈ 0.109, whereas the conditional probability that the second digit is 2, *given* than the first digit is 1, is ≈ 0.115. Again the situation for squares $\{k^2\}$ is different. Back in 1935, the distinguished Russian-British mathematician A. S. Besicovitch [**3**] showed that the *totality* of the digits in the squares are uniformly distributed, in the following sense. Let $\epsilon > 0$. A positive integer, whose decimal representation is $d_1 d_2 d_3 \cdots d_\ell$, is *$\epsilon$-normal* if for $d = 0, 1, 2, 3, \ldots, 9$, we have

$$\left| \frac{1}{\ell} \# \left\{ j : j \le \ell \quad \text{and} \quad d_j = d \right\} - \frac{1}{10} \right| < \epsilon.$$

Besicovitch's main theorem states that, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ m : m \le n \quad \text{and} \quad m^2 \text{ is } \epsilon\text{-normal} \right\} = 1.$$

Suggested undergraduate research

1. Give a simple proof that $\lim_{n\to\infty} \frac{1}{n}\#\{k : k \le n \text{ and } D(k) = d\}$ does not exist for $d = 1, 2, \ldots, 9$. For which $r > 0$ do the limits $\lim_{n\to\infty} \frac{1}{n}\#\{k : k \le n \text{ and } D(kr) = d\}$ exist for $d = 1, 2, \ldots, 9$?

2. Prove equation (5). It's hard to believe, given Theorem 2, that this is a coincidence. Try to find a different proof, which involves first digits of sequences.
3. As noted in [**8**], some rapidly-increasing sequences satisfy

$$\lim_{n\to\infty} \frac{1}{n}\#\{k : k \le n \quad \text{and} \quad D(a_k) = d\} = \log_{10}(d+1) - \log_{10} d \tag{10}$$

   for $d = 1, 2, \ldots, 9$. This is true for the Fibonacci sequence, because it is almost an exponential sequence; see [**7**] for the general result. This is also true for $a_k = k!$ by [**4**, Theorem 3].
   (a) Find more examples; a good source for sequences is [**9**].
   (b) Find some general conditions that are sufficient to imply assertion (10). This might be difficult, because it isn't sufficient to assume that $\{a_k\}$ grows exponentially or faster. An obvious counterexample is $\{10^k\}$; another is $\{(10 + \frac{10}{k})^k\}$.
4. Does (10) hold for every sequence $\{p(k)\}$, where p is a polynomial of degree at least 2 that is positive on $[a, \infty)$ for some a? See [**4**, Corollary 2].

Acknowledgments It is a pleasure to thank Steven R. Finch, of the Statistics Department at Harvard University, for alerting me to Werner Hürlimann's paper and for his subsequent interest in this project. I also thank James D. Harper, at Central Washington University, for helping me make the proof of Theorem 4 more friendly.

REFERENCES

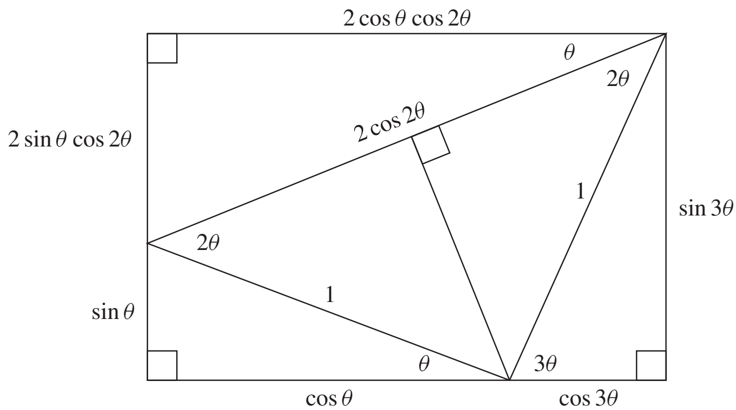
1. A. Berger and T. P. Hill, Benford's law strikes back: No simple explanation in sight for mathematical gem, *Math. Intelligencer* **33** (2011) 85–91. http://dx.doi.org/10.1007/s00283-010-9182-3.
2. A. Berger and T. P. Hill, Benford Online Bibliography, available at http://www.benfordonline.net.
3. A. S. Besicovitch, The asymptotic distribution of the numerals in the decimal representation of the squares of the natural numbers, *Math. Z.* **39** (1935) 146–156.
4. P. Diaconis, The distribution of leading digits and uniform distribution mod 1, *Ann. Probab.* **5** (1977) 72–81. http://dx.doi.org/10.1214/aop/1176995891.
5. T. P. Hill, Base-invariance implies Benford's law, *Proc. Amer. Math. Soc.* **123** (1995) 887–895.
6. W. Hürlimann, Integer powers and Benford's law, *Internat. J. Pure Applied Math.* **11** (2004) 39–46.
7. T. M. Katz and D. I. A. Cohen, The first digit property for exponential sequences is independent of the underlying distribution, *Fibonacci Quarterly* **24** (1986) 2–7.
8. K. A. Ross, Benford's Law, a growth industry, *Amer. Math. Monthly* **118** (2011) 571–583. http://dx.doi.org/10.4169/amer.math.monthly.118.07.571.
9. N. J. A. Sloane, *A Handbook of Sequences*, Academic Press, New York, 1973.
10. K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

Summary Benford's Law is the observation that in many lists of numbers that arise in the real world, for $d = 1, 2, \ldots, 9$, the likelihood that the first digit is d is $\log_{10}((d+1)/d)$. A similar phenomenon is known for essentially all geometric sequences ar^k, namely the relative frequencies $R_N(d)$ of the first digit d of the first N terms in such a sequence tend to $\log_{10}((d+1)/d)$. In this note, it is shown that the analogous statements for the sequence of squares and the sequence of cubes do not hold. However, in these cases, interesting subsequences of $R_N(d)$ do converge, but not to $\log_{10}((d+1)/d)$.

Proof Without Words: The Triple Angle Sine and Cosine Formulas

CLAUDI ALSINA
Universitat Politècnica de Catalunya
Barcelona, Spain

ROGER B. NELSEN
Lewis & Clark College
Portland, Oregon



$$\begin{aligned}\sin 3\theta &= 2\sin\theta\cos 2\theta + \sin\theta \\ &= 2\sin\theta\,(1-2\sin^2\theta)+\sin\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

$$\begin{aligned}\cos 3\theta &= 2\cos\theta\cos 2\theta - \cos\theta \\ &= 2\cos\theta\,(2\cos^2\theta-1)-\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

REFERENCE

S. Okuda, Proof without words: The triple-angle formulas for sine and cosine, *Math. Mag.* **74** (2001) 135.

Math. Mag. **85** (2012) 43. doi:10.4169/math.mag.85.1.43.

Cyclotomic Polynomials, Symmetric Polynomials, and a Generalization of Euler's Totient Function

JULIAN FREED-BROWN
Department of Physics
University of Chicago
Chicago, IL 60637
jfreedbrown@uchicago.edu

MATTHEW HOLDEN
Google, Inc.
Mountain View, CA 94043
holden.matt@gmail.com

MICHAEL E. ORRISON
Harvey Mudd College
Claremont, CA 91711
orrison@hmc.edu

MICHAEL VRABLE
Google, Inc.
Mountain View, CA 94043
vrable@cs.hmc.edu

For every positive integer n, Euler's totient function, or ϕ-function, gives the number $\phi(n)$ of positive integers less than n that are relatively prime to n, with the convention that $\phi(1) = 1$. Students of abstract algebra also know $\phi(n)$ as the number of generators of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. It therefore seems worthwhile to consider generalizations of Euler's totient function from a group theoretic perspective.

One such generalization is Jordan's totient function $J_k(n)$, which is defined for all positive integers k and n. We describe this function in the first section, and find that it generalizes Euler's totient function in that $J_1(n) = \phi(n)$. We then use our group theoretic approach to establish a recursive formula for $J_k(n)$.

In this note, we introduce another generalization of Euler's totient function, which we denote by $M_k(n)$. It resembles Jordan's totient function in that $M_1(n) = \phi(n)$, and it obeys a recursion similar to the recursion for the Jordan totient function. Using this recursion, we find that $M_k(n)$ can be described by a polynomial in the prime factors of n. Furthermore, for certain choices of n and k, these polynomials have interesting and sometimes unexpected structure. In particular, as we show, cyclotomic polynomials and complete homogeneous symmetric polynomials (which we define below) appear as factors when n has at most two distinct prime divisors.

Jordan's Totient Function, $J_k(n)$

Jordan's totient function is described, for example, in [**1**, pp. 147–155]). For positive integers n and k, $J_k(n)$ is defined to be the number of k-tuples $(a_1, \ldots, a_k)$ from $\{1, \ldots, n\}$ such that the greatest common divisor of $\{a_1, \ldots, a_k\}$ is relatively prime

Math. Mag. **85** (2012) 44–50. doi:10.4169/math.mag.85.1.44. © Mathematical Association of America

to n. Jordan's totient function is a generalization Euler's totient function because $J_1(n) = \phi(n)$.

To view J_k from a group theoretic perspective, note that $J_k(n)$ counts the number of sequences $(g_1, \ldots, g_k)$ of elements in $\mathbb{Z}/n\mathbb{Z}$ such that, if G_i is the subgroup generated by $\{g_1, \ldots, g_i\}$, then

$$\{0\} \leq G_1 \leq \cdots \leq G_{k-1} \leq G_k = \mathbb{Z}/n\mathbb{Z}.$$

Furthermore, by using simple properties of subgroups and quotient groups of $\mathbb{Z}/n\mathbb{Z}$, identities involving $J_k(n)$ may be obtained. For example, Gegenbauer (see [**1**, p. 151]) showed that

$$J_{k+\ell}(n) = \sum_{\substack{1\leq d\leq n\\ d\mid n}} d^\ell J_k(d) J_\ell(n/d)$$

for all positive integers n, k, and ℓ.

To see why Gegenbauer's result is true, recall that for each divisor d of n, there is a unique subgroup of order d in $\mathbb{Z}/n\mathbb{Z}$. Furthermore, the corresponding quotient group $(\mathbb{Z}/n\mathbb{Z})/(\mathbb{Z}/d\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(n/d)\mathbb{Z}$. There are $J_k(d)$ sequences of length k for $\mathbb{Z}/d\mathbb{Z}$. Every extension of such a sequence to a sequence of length $k + \ell$ for $\mathbb{Z}/n\mathbb{Z}$ corresponds to a sequence of length ℓ for the quotient group. Because every element of the quotient group has d representatives in $\mathbb{Z}/n\mathbb{Z}$, the number of sequences of length $k + \ell$ that pass through $\mathbb{Z}/d\mathbb{Z}$ is $d^\ell J_k(d) J_\ell(n/d)$. Summing over all divisors gives Gegenbauer's result. (See [**1**] or [**5**] for other identities involving $J_k(n)$, and [**4**] for many other generalizations of Euler's totient function.)

A new totient function, $M_k(n)$

In this note, we consider a generalization of Euler's totient function that is similar in spirit to Jordan's totient function. For positive integers n and k, define $M_k(n)$ to be the number of sequences $(g_1, \ldots, g_k)$ of elements in $\mathbb{Z}/n\mathbb{Z}$ such that, if G_i is the subgroup generated by $\{g_1, \ldots, g_i\}$, then

$$\{0\} < G_1 < \cdots < G_{k-1} < G_k = \mathbb{Z}/n\mathbb{Z}.$$

In other words, $M_k(n)$ counts only those sequences $(g_1, \ldots, g_k)$ with the property that G_i is strictly contained in G_{i+1} for $i = 1, \ldots, k-1$. Together with the convention that $M_1(1) = 1$, we have that $M_1(n) = \phi(n)$. The function $M_k(n)$ is therefore another generalization of Euler's totient function.

A noteworthy feature of $M_k(n)$ is that, for a fixed n, $M_k(n)$ will eventually become 0. In fact, if $n = p_1^{e_1} \cdots p_r^{e_r}$ where the p_i are prime, then $M_k(n) = 0$ for all $k > e_1 + \cdots + e_r$. This is because there are no appropriate sequences of subgroups of length more than $e_1 + \cdots + e_r$. Also, unlike Jordan's totient function, M_k is not multiplicative when $k > 1$, i.e., $M_k(m)M_k(n)$ need not equal $M_k(mn)$ when m and n are relatively prime. For example, $M_2(6) = 10$ while $M_2(2) = M_2(3) = 0$. Nonetheless, the group theoretic definition of M_k allows us to see interesting relationships between the values of $M_k(n)$ for different choices of k and n. Consider, for example, the following theorem, which is an analogue to Gegenbauer's theorem.

THEOREM 1. *If n, k and ℓ are positive integers, then*

$$M_{k+\ell}(n) = \sum_{\substack{1<d<n\\ d\mid n}} d^\ell M_k(d) M_\ell(n/d).$$

Proof. The argument is similar to that for Gegenbauer's result, the only change being that the sum is now over the nontrivial divisors of n (that is, $d \neq 1$ and $d \neq n$) due to the strict containment of the corresponding subgroups. ■

By recursively applying Theorem 1 and using the fact that $M_1 = \phi$ is multiplicative, it should be clear that $M_k(n)$ can be expressed as a polynomial in the prime divisors of n. For example, if p and q are distinct primes, and $e \geq 1$, then $M_1(p^e) = \phi(p^e) = (p-1)p^{(e-1)}$, $M_2(p^3) = p^3(p-1)^2 + p^2(p-1)^2 = (p-1)^2 p^2(p+1)$, and

$$M_5(p^3q^2) = (p-1)^3 p^3 (q-1)^2 q \left(p^2+q^2\right)\left(p^4+p^3q+p^2q^2+pq^3+q^4\right).$$

The Prime Power Case

In this section, we compute $M_k(p^e)$, where p is prime and k and e are positive integers. We begin with two lemmas.

LEMMA. *If n, k, and e are positive integers, and p is prime, then*

$$M_{k+1}(p^e) = (p-1)p^{e-1}\sum_{j=k}^{e-1} M_k(p^j).$$

Proof. By Theorem 1,

$$M_{k+1}(p^e) = \sum_{\substack{1<d<p^e \\ d\mid p^e}} dM_k(d)M_1(p^e/d) = \sum_{j=1}^{e-1} p^j M_k(p^j)M_1(p^{e-j}).$$

If $k > j$, then $M_k(p^j) = 0$. Also, $M_1(p^{e-j}) = \phi(p^{e-j}) = (p-1)p^{e-j-1}$. Thus,

$$M_{k+1}(p^e) = \sum_{j=k}^{e-1}(p-1)p^{e-1}M_k(p^j) = (p-1)p^{e-1}\sum_{j=k}^{e-1} M_k(p^j).$$ ■

LEMMA. *If p is prime and $e \geq k \geq 1$, then*

$$(p^{k-1}-1)\sum_{j=k-1}^{e-1}\left(p^{j-(k-1)}\prod_{i=1}^{k-2}(p^{j-i}-1)\right) = \prod_{i=1}^{k-1}(p^{e-i}-1).$$

Proof. We prove this by induction on $e-k$. First, note that if $e-k=0$, then

$$(p^{e-1}-1)\sum_{j=e-1}^{e-1}\left(p^{j-(e-1)}\prod_{i=1}^{e-2}(p^{j-i}-1)\right) = \prod_{i=1}^{e-1}(p^{e-i}-1).$$

This agrees with the formula. Now let $e-k \geq 1$ and assume that the formula holds for $e-k-1$. Then

$$(p^{k-1}-1)\sum_{j=k-1}^{e-1}\left(p^{j-(k-1)}\prod_{i=1}^{k-2}(p^{j-i}-1)\right)$$
$$= \prod_{i=1}^{k-1}(p^{e-1-i}-1) + (p^{k-1}-1)p^{e-k}\prod_{i=1}^{k-2}(p^{e-1-i}-1)$$

$$= \left(\prod_{i=1}^{k-2}(p^{e-1-i}-1)\right)\left((p^{e-k}-1)+(p^{k-1}-1)p^{e-k}\right)$$

$$= \prod_{i=1}^{k-1}(p^{e-i}-1). \qquad \blacksquare$$

Recall that if $k > e$, then $M_k(p^e) = 0$. Using the two lemmas, we can now give a formula for $M_k(p^e)$ when $e \geq k$. In the next section, we restate this formula using cyclotomic polynomials.

THEOREM 2. *If p is prime and $e \geq k \geq 1$, then*

$$M_k(p^e) = (p-1)^k p^{e+\frac{k(k-3)}{2}} \prod_{i=1}^{k-1} \frac{p^{e-i}-1}{p^i-1}.$$

Proof. We prove this by induction on k. First, note that if $k = 1$, then

$$M_1(p^e) = \phi(p^e) = (p-1)p^{e-1}.$$

This agrees with the formula. Now let $k \geq 2$ and assume that the formula holds for all positive integers less than k. Using the two lemmas we have

$$\begin{aligned}
M_k(p^e) &= (p-1)p^{e-1} \sum_{j=k-1}^{e-1} M_{k-1}(p^j) \\
&= (p-1)p^{e-1} \sum_{j=k-1}^{e-1} \left((p-1)^{k-1} p^{j+\frac{(k-1)(k-4)}{2}} \prod_{i=1}^{k-2} \frac{p^{j-i}-1}{p^i-1}\right) \\
&= (p-1)^k p^{e+\frac{k(k-3)}{2}} \sum_{j=k-1}^{e-1} \left(p^{j-(k-1)} \prod_{i=1}^{k-2} \frac{p^{j-i}-1}{p^i-1}\right) \\
&= (p-1)^k p^{e+\frac{k(k-3)}{2}} \left(\prod_{i=1}^{k-1} \frac{1}{p^i-1}\right) \\
&\qquad (p^{k-1}-1) \sum_{j=k-1}^{e-1} \left(p^{j-(k-1)} \prod_{i=1}^{k-2} (p^{j-i}-1)\right) \\
&= (p-1)^k p^{e+\frac{k(k-3)}{2}} \left(\prod_{i=1}^{k-1} \frac{1}{p^i-1}\right) \prod_{i=1}^{k-1}(p^{e-i}-1) \\
&= (p-1)^k p^{e+\frac{k(k-3)}{2}} \prod_{i=1}^{k-1} \frac{p^{e-i}-1}{p^i-1}. \qquad \blacksquare
\end{aligned}$$

Cyclotomic polynomials

Let $d \geq 1$, and let $\Phi_d(x)$ denote the dth *cyclotomic polynomial*, which is a degree $\phi(d)$ polynomial whose roots are the primitive dth roots of unity. In other words, the roots of $\Phi_d(x)$ are those complex numbers ω with the property that $\omega^d = 1$, while $\omega^c \neq 1$ for all c such that $1 \leq c < d$.

For example, $\Phi_1(x) = x - 1$, $\Phi_2(x) = x + 1$, $\Phi_3(x) = x^2 + x + 1$, and $\Phi_4(x) = x^2 + 1$. In particular, note that the roots of $\Phi_4(x)$ are the primitive fourth roots of unity, i and $-i$. Also, note that if $n \geq 1$, then $x^n - 1 = \prod_{d|n} \Phi_d(x)$. We take advantage of this fact in our proof of the corollary below. (See, for example, Section 13.6 in [**2**] for an introduction to cyclotomic polynomials.)

Let e and k be positive integers where $e \geq k$. Let $S_{e,k}$ be the multiset of divisors of $e-1, e-2, \ldots, e-(k-1)$ after removing the multiset of divisors of $1, 2, \ldots, k-1$. For example,

$$S_{10,3} = \{9, 3, 1, 8, 4, 2, 1\} - \{1, 1, 2\} = \{9, 8, 4, 3\}.$$

We may now rewrite the formula in Theorem 2 using cyclotomic polynomials.

COROLLARY. *If p is prime and $e \geq k \geq 1$, then*

$$M_k(p^e) = (p-1)^k p^{e+\frac{k(k-3)}{2}} \prod_{i \in S_{e,k}} \Phi_i(p).$$

Proof. By Theorem 2,

$$\begin{aligned} M_k(p^e) &= (p-1)^k p^{e+\frac{k(k-3)}{2}} \prod_{i=1}^{k-1} \frac{p^{e-i}-1}{p^i-1} \\ &= (p-1)^k p^{e+\frac{k(k-3)}{2}} \prod_{i=1}^{k-1} \frac{\prod_{d|(e-i)} \Phi_d(p)}{\prod_{d'|i} \Phi_{d'}(p)} \\ &= (p-1)^k p^{e+\frac{k(k-3)}{2}} \prod_{i \in S_{e,k}} \Phi_i(p). \end{aligned}$$ ■

For example, if p is prime, then

$$\begin{aligned} M_3(p^{10}) &= (p-1)^3 p^{10} \left(p^2+p+1\right)\left(p^2+1\right)\left(p^4+1\right)\left(p^6+p^3+1\right) \\ &= (p-1)^3 p^{10} \Phi_3(p)\Phi_4(p)\Phi_8(p)\Phi_9(p). \end{aligned}$$

The Two-Prime-Powers Case for maximal subgroup chains

Let $M(n) = M_k(n)$ where k is the largest integer such that $M_k(n)$ is nonzero. In other words, if the prime factorization of n is $p_1^{e_1} \cdots p_m^{e_m}$, then $k = e_1 + \cdots + e_m$. For example, if p and q are distinct primes, then

$$\begin{aligned} M(p^4q^2) &= M_6(p^4q^2) \\ &= (p-1)^4 p^6 (q-1)^2 q \left(p^2 - pq + q^2\right) \\ &\qquad \left(p^2+pq+q^2\right)\left(p^4+p^3q+p^2q^2+pq^3+q^4\right). \end{aligned}$$

Note that the factor

$$\left(p^2 - pq + q^2\right)\left(p^2+pq+q^2\right)\left(p^4+p^3q+p^2q^2+pq^3+q^4\right)$$

in $M(p^4q^2)$ is a symmetric polynomial in p and q. As we show below in Theorem 3, such symmetric polynomials in p and q appear in every polynomial of the form $M(p^eq^f)$.

Let x and y be distinct indeterminates. Let $k \geq 1$, and define

$$\begin{aligned} h_k(x, y) &= x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + x^2y^{k-2} + xy^{k-1} + y^k \\ &= \frac{x^{k+1} - y^{k+1}}{x - y}. \end{aligned}$$

Define $h_0(x, y) = 1$. The polynomial $h_k(x, y)$ is called the *complete homogeneous symmetric polynomial* of degree k in x and y. (See, for example, Chapter 7 of [**6**] for more about symmetric polynomials.) In the proof of the following theorem, we use the elementary fact that

$$y^f h_{e-1}(x, y) + x^e h_{f-1}(x, y) = h_{e+f-1}(x, y). \tag{1}$$

THEOREM 3. *If p and q are distinct primes, and e and f are nonnegative integers, then*

$$M(p^e q^f) = (p-1)^e (q-1)^f p^{\frac{e(e-1)}{2}} q^{\frac{f(f-1)}{2}} \prod_{i=0}^{f-1} \frac{h_{e+i}(p, q)}{h_i(p, q)}.$$

Proof. We prove this by induction on the sum $e + f$. Note that, when $e + f = 0$, the formula holds trivially because $M(p^0q^0) = 1$. Now let $e + f \geq 1$ and assume that the formula holds for $M(p^e q^{f-1})$ and $M(p^{e-1}q^f)$. Then, by Theorem 1,

$$\begin{aligned} M(p^e q^f) &= (p-1)p^{e-1}q^f M(p^{e-1}q^f) + (q-1)p^e q^{f-1} M(p^e q^{f-1}) \\ &= (p-1)p^{e-1}q^f \left((p-1)^{e-1}(q-1)^f p^{\frac{(e-1)(e-2)}{2}} q^{\frac{f(f-1)}{2}} \prod_{i=0}^{f-1} \frac{h_{e-1+i}(p, q)}{h_i(p, q)} \right) \\ &\quad + (q-1)p^e q^{f-1} \left((p-1)^e (q-1)^{f-1} p^{\frac{e(e-1)}{2}} q^{\frac{(f-1)(f-2)}{2}} \prod_{i=0}^{f-2} \frac{h_{e+i}(p, q)}{h_i(p, q)} \right) \\ &= (p-1)^e (q-1)^f p^{\frac{e(e-1)}{2}} q^{\frac{f(f-1)}{2}} \left(q^f \prod_{i=1}^{f-1} \frac{h_{e-1+i}(p, q)}{h_i(p, q)} + p^e \prod_{i=0}^{f-2} \frac{h_{e+i}(p, q)}{h_i(p, q)} \right). \end{aligned}$$

By (1), this then becomes

$$\begin{aligned} M(p^e q^f) &= (p-1)^e (q-1)^f p^{\frac{e(e-1)}{2}} q^{\frac{f(f-1)}{2}} \\ &\qquad \left(\frac{q^f h_{e-1}(p, q) + p^e h_{f-1}(p, q)}{h_{f-1}(p, q)} \right) \prod_{i=0}^{f-2} \frac{h_{e+i}(p, q)}{h_i(p, q)} \\ &= (p-1)^e (q-1)^f p^{\frac{e(e-1)}{2}} q^{\frac{f(f-1)}{2}} \prod_{i=0}^{f-1} \frac{h_{e+i}(p, q)}{h_i(p, q)}. \end{aligned}$$

■

Because of the symmetric relationship between the primes p and q in Theorem 3, we also have the following corollary, a direct proof of which is relatively straightforward.

COROLLARY. *If e and f are nonnegative integers, then*

$$\prod_{i=0}^{f-1} \frac{h_{e+i}(x, y)}{h_i(x, y)} = \prod_{i=0}^{e-1} \frac{h_{f+i}(x, y)}{h_i(x, y)}.$$

Possible next steps

It would be interesting to extend the definition of M_k to include other finite groups. This has already been done for J_k (see, for example, Hall's influential paper [**3**]). Some preliminary work suggests that there may be interesting patterns for certain classes and sizes of finite groups. Finally, we noticed that if $k \geq 1$, then $J_k(4) = M_2(2^{k+1})$, where both values are equal to $2^k(2^k - 1)$. Given the group theoretic definitions of J_k and M_k, it would be interesting to find a combinatorial proof of this identity.

Acknowledgment Special thanks to Gary Brookfield, Lenny Fukshansky, Shahriar Shahriari, and Francis Su for helpful conversations.

REFERENCES

1. L. Dickson, *History of the Theory of Numbers, Vol. I: Divisibility and Primality*, Chelsea, New York, 1966.
2. D. Dummit and R. Foote, *Abstract Algebra*, 3rd ed., Wiley, Hoboken, NJ, 2004.
3. P. Hall, The Eulerian functions of a group, *Quart. J. Math. Oxford Ser.* 7 (1936) 134–151. http://dx.doi.org/10.1093/qmath/os-7.1.134.
4. J. Sándor and B. Crstici, *Handbook of Number Theory, II*, Kluwer, Dordrecht, 2004.
5. R. Sivaramakrishnan, The many facets of Euler's totient. II. Generalizations and analogues, Nieuw Arch. Wisk. (4) **8**(2) (1990) 169–187.
6. R. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.

Summary We introduce a generalization of Euler's totient function that, when applied to an integer $n \geq 2$, can be written as a polynomial in the prime factors of n. We then show how cyclotomic and complete homogeneous symmetric polynomials appear as factors of these polynomials when n has at most two distinct prime divisors.

Math Bite: Finding e in Pascal's Triangle

HARLAN J. BROTHERS
Brothers Technology, LLC
Branford, CT 06405
harlan@brotherstechnology.com

Mathematicians have long been familiar with the tidy way in which the nth row of Pascal's triangle sums to 2^n (the top row conventionally labeled as $n = 0$). It is less obvious how the rows behave when we multiply their items.

1	1
1 1	1
1 2 1	2
1 3 3 1	9
1 4 6 4 1	96
1 5 10 10 5 1	2500
1 6 15 20 15 6 1	162000

Let s_n be the product for row n; that is, $s_n = \prod_{k=0}^{n} \binom{n}{k}$. On the right-hand side of the figure above, we see the sequence $\{s_n\}$ grows very quickly. To get a sense of its rate of growth, we can look at the ratios of successive terms, $r_n = s_n/s_{n-1}$. The sequence $\{r_n\}$ itself grows rapidly. Examining the ratios of ratios, a familiar pattern emerges:

n	s_n	$r_n = s_n/s_{n-1} \approx$	$r_n/r_{n-1} \approx$
1	1	1	
2	2	2	2
3	9	4.5	2.25
4	96	10.667	2.370
5	2500	26.042	2.441
6	162000	64.800	2.488
⋮	⋮	⋮	⋮
1000	$\approx 1.68\ 10^{215681}$	$2.49\ 10^{432}$	2.71692

THEOREM. $\lim_{n\to\infty} \dfrac{s_{n+1}/s_n}{s_n/s_{n-1}} = e.$

Proof. By direct calculation we get

$$s_n = (n!)^{n+1} \prod_{k=0}^{n} (k!)^{-2}, \quad n \geq 0$$

$$s_n/s_{n-1} = \frac{n^n}{n!}, \quad n \geq 1, \quad \text{and}$$

$$\frac{s_{n+1}/s_n}{s_n/s_{n-1}} = \left(1 + \frac{1}{n}\right)^n.$$

Given that $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$, the result follows. ■

Summary If s_n is the product of the entries in row n of Pascal's triangle then $(s_{n+1}/s_n)/(s_n/s_{n-1}) = (1 + 1/n)^n$, which has the limiting value e.

Math. Mag. **85** (2012) 51.doi:10.4169/math.mag.85.1.51.

Plausible and Genuine Extensions of L′Hospital′s Rule

J. MARSHALL ASH
DePaul University
Chicago, IL 60614
MASH@depaul.edu

ALLAN BERELE
DePaul University
Chicago, IL 60614
aberele@condor.depaul.edu

STEFAN CATOIU
DePaul University
Chicago, IL 60614
scatoiu@condor.depaul.edu

A plausible extension

Roughly speaking, L'Hospital's Rule says that if $f(x)/g(x)$ is indeterminate at infinity and if x is large, then $f(x)/g(x)$ is approximately equal to $f'(x)/g'(x)$. Also, the limit comparison test says that if a_n is approximately equal to b_n then $\sum a_n$ converges if and only if $\sum b_n$ converges. The best thing that one could possibly hope for in trying to combine these two observations is the equivalence

$$\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} \text{ converges if and only if } \sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \text{ converges.} \tag{1}$$

Indeed this is true for certain classes of functions, including non-constant polynomials and exponential functions. L'Hospital's Rule as stated above is an implication, so it seems more prudent to predict that the convergence of $\sum \frac{f'(n)}{g'(n)}$ might imply the convergence of $\sum \frac{f(n)}{g(n)}$. In order to formulate a reasonable conjecture we first need the precise statement of L'Hospital's Rule [**5, 6, 7**].

We will say that two real valued functions f, g of a real variable generate an indeterminate form $0/0$ at infinity if

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0; \tag{2}$$

and an indeterminate form ∞/∞ at infinity in any of the following four cases:

$$\lim_{x\to\infty} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = \pm\infty.$$

One standard version of L'Hospital's Rule asserts that if f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and if

$$g'(x) \neq 0 \text{ in some neighborhood of } \infty, \tag{3}$$

then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \text{ implies } \lim_{x\to\infty} \frac{f(x)}{g(x)} = L \tag{4}$$

where L is an extended real number.

Math. Mag. **85** (2012) 52–60. doi:10.4169/math.mag.85.1.52.

PLAUSIBLE CONJECTURE. *If f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and satisfy* (3), *and if $\sum f'(n)/g'(n)$ converges, then so does $\sum f(n)/g(n)$.*

As with L'Hospital's Rule itself, the reverse implication

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \text{ converges implies } \sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} \text{ converges}$$

is not always true. For example, if we let

$$f(x) = \frac{\sin \pi x}{x} \quad \text{and} \quad g(x) = \frac{1}{x},$$

then f, g generate the indeterminate form $0/0$ at infinity. But for every integer n, $\sin \pi n = 0$ and $\cos \pi n = (-1)^n$, so

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} = \sum_{n=1}^{\infty} \frac{0/n}{1/n} \text{ converges,}$$

while $f'(x) = \dfrac{-\sin \pi x}{x^2} + \dfrac{\pi \cos \pi x}{x}$, $f'(n) = \dfrac{\pi\,(-1)^n}{n}$, $g'(n) = -\dfrac{1}{n^2}$, so that

$$\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} = \sum_{n=1}^{\infty} \frac{\pi(-1)^n/n}{-1/n^2} = \sum_{n=1}^{\infty} \pi n(-1)^{n+1} \text{ diverges.}$$

Again, as in L'Hospital's Rule we need hypothesis (3), as is demonstrated by the following example, adapted from [**1**]. Let

$$f(x) = 6\pi x + \sin 6\pi x,$$
$$g(x) = e^{\sin 3\pi x} f(x).$$

Then

$$\frac{f'(x)}{g'(x)} = \frac{6\pi + 6\pi \cos 6\pi x}{3\pi(\cos 3\pi x)e^{\sin 3\pi x} f(x) + e^{\sin 3\pi x} f'(x)}$$
$$= \left(\frac{2}{e^{\sin 3\pi x}}\right) \frac{\cos 6\pi x + 1}{\cos 3\pi x(6\pi x) + \cos 3\pi x(\sin 6\pi x) + 2 + 2\cos 6\pi x}.$$

Hence, if n is an integer

$$\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{3\pi n + 2(-1)^n},$$

which converges since its terms are alternating and decreasing to zero. (Note that we used 3π instead of π to get increasing denominators.) Nevertheless,

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} = \sum_{n=1}^{\infty} e^{-\sin 3\pi n} = \sum_{n=1}^{\infty} 1 = \infty.$$

Our main result is that the Plausible Conjecture is true only under certain conditions. One surprise (at least to us) is that the relationship between $\sum \frac{f(n)}{g(n)}$ and $\sum \frac{f'(n)}{g'(n)}$ is different in the $0/0$ and ∞/∞ cases, a distinction that does not occur in L'Hospital's Rule.

Two examples with $\sum \frac{f'(n)}{g'(n)}$ convergent and $\sum \frac{f(n)}{g(n)}$ divergent

EXAMPLE 1. The first example has f, g generating the indeterminate form $0/0$ at infinity, $g'(x) \neq 0$ near infinity, $\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)}$ convergent, and $\sum_{n=1}^{\infty} \frac{f(n)}{g(n)}$ divergent.

The idea is that $f(n)$ and $f'(n)$ may be chosen at will, subject only to the condition

$$\lim_{n\to\infty} f(n) = 0.$$

So we define for each natural number n,

$$f(n) = \frac{1}{n} \quad \text{and} \quad f'(n) = 0$$

and on each interval of the form $[n, n+1]$ let f be the unique cubic function satisfying these four boundary conditions:

$$f(n) = \frac{1}{n}, \quad f'(n) = 0, \quad f(n+1) = \frac{1}{n+1}, \quad f'(n+1) = 0.$$

Since every cubic with two critical points is monotone on the interval between them, we have for all x in $[n, n+1]$,

$$\frac{1}{n} \geq f(x) \geq \frac{1}{n+1},$$

so that $\lim_{x\to\infty} f(x) = 0$. On the interval $[1, 5]$, f looks like FIGURE 1.

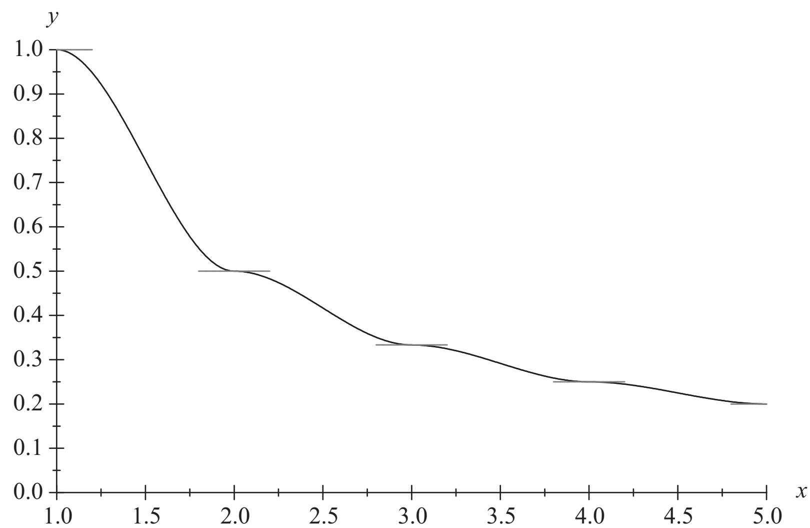


Figure 1 The gray horizontal tangent line segments show that $f'(x)$ is zero when x is an integer.

For $g(x)$, we choose $g(x) = \frac{1}{x}$ (for all $x > 0$) so that $g'(x) = -\frac{1}{x^2}$. Then we have

$$\sum \frac{f'(n)}{g'(n)} = \sum \frac{0}{-1/n^2} = \sum 0, \text{ which converges, and}$$

$$\sum \frac{f(n)}{g(n)} = \sum \frac{1/n}{1/n} = \sum 1, \text{ which diverges,}$$

and we are done.

A formula for the function $f(x)$ is

$$f(x) = \frac{1}{\lfloor x \rfloor} - \frac{(x - \lfloor x \rfloor)^2 (3 - 2(x - \lfloor x \rfloor))}{\lfloor x \rfloor (\lfloor x \rfloor + 1)}.$$

We leave as exercises for the reader to show that this example would have the same properties if $f(x)$ were replaced by $\frac{\pi}{2} - \mathrm{Si}(2\pi x)$ where $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$.

EXAMPLE 2. The second example has f, g generating the indeterminate form ∞/∞ at infinity, $g'(x) \neq 0$ near infinity, $\frac{f'(x)}{g'(x)}$ decreasing near infinity, $\sum_{n=2}^{\infty} \frac{f'(n)}{g'(n)}$ convergent, and $\sum_{n=2}^{\infty} \frac{f(n)}{g(n)}$ divergent.

Let

$$f(x) = \ln x,$$
$$g(x) = x \ln^2 x - 2x \ln x + 2x \sim x \ln^2 x$$

so that

$$f'(x) = \frac{1}{x} \quad \text{and}$$
$$g'(x) = \ln^2 x.$$

Then

$$\sum_{n=2}^{\infty} \frac{f'(n)}{g'(n)} = \sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \text{ converges,}$$

while by the limit comparison test

$$\sum_{n=2}^{\infty} \frac{f(n)}{g(n)} \text{ diverges since } \sum_{n=2}^{\infty} \frac{\ln n}{n \ln^2 n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

Genuine extensions

The main purpose of counterexamples is to point the way to positive results. In this section we will present two of them.

The first theorem applies when the indeterminant form is $0/0$. Example 2 shows that the conclusion need not hold when $0/0$ is replaced by ∞/∞.

THEOREM 1. *Let f and g be differentiable functions on $(0, \infty)$ such that f, g generate the indeterminate form $0/0$ at infinity, $g'(x) \neq 0$ in a neighborhood of infinity, and $g(n)$ and $g'(n)$ are nonzero for all $n \in \mathbb{N}$. If*

$$\sum_{n=1}^{\infty} \sup_{x \geq n} \left| \frac{f'(x)}{g'(x)} \right| \textit{ converges,}$$

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \textit{ converges absolutely.}$$

Proof. This is immediate once we know that from differentiability, (3), and (2) there follows this generalization of the Cauchy mean value theorem: For each $n \geq 1$, there is a $c = c_n > n$ such that

$$\left|\frac{f(n)}{g(n)}\right| = \left|\frac{f'(c)}{g'(c)}\right| \leq \sup_{x \geq n} \left|\frac{f'(x)}{g'(x)}\right|. \qquad \blacksquare$$

When $|f'(x)/g'(x)|$ is decreasing in a neighborhood of infinity, we can replace the convergence of the series of suprema by the convergence of

$$\sum_{n=1}^{\infty} \left|\frac{f'(n)}{g'(n)}\right|,$$

since in that case $\left|\frac{f'(n)}{g'(n)}\right| = \sup_{x \geq n} \left|\frac{f'(x)}{g'(x)}\right|$ for sufficiently large n.

We ask the reader to contrast Theorem 1 with Example 2. The example shows that if we change $0/0$ to ∞/∞ in the hypothesis, the desired conclusion (that $\sum f(n)/g(n)$ converges) may no longer follow. This is the distinction alluded to at the end of the first section.

Also, we can can identify a class of functions for which the original equivalence (1) holds in its entirety. This class includes polynomials, and a much wider class of functions as well.

THEOREM 2. *Let f and g be differentiable functions such that there are nonzero integers i and j and nonzero real numbers a and b and*

$$f(x) = ax^i + o(x^i) \qquad g(x) = bx^j + o(x^j)$$
$$f'(x) = iax^{i-1} + o(x^{i-1}) \qquad g'(x) = jbx^{j-1} + o(x^{j-1}).$$

Then if f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity and $g(n)$ and $g'(n)$ are nonzero for all $n \in \mathbb{N}$, then equivalence (1) is true.

Proof. The condition $j \geq i + 2$ is necessary and sufficient for the convergence of both $\sum \frac{f(n)}{g(n)}$ and $\sum \frac{f'(n)}{g'(n)}$. $\blacksquare$

COROLLARY. *Let f and g be functions analytic in $\mathbb{C} \setminus \{0\}$ and not having an essential singularity at infinity. If f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and $g(n)$ and $g'(n)$ are not zero for all $n \in \mathbb{N}$; then equivalence (1) is true.*

Proof. The first hypotheses means that we may write

$$f(x) = \sum_{\nu=-\infty}^{i} a_\nu x^\nu \qquad g(x) = \sum_{\nu=-\infty}^{j} b_\nu x^\nu$$

where a_i and b_j are nonzero. If $0/0$ is generated, then $i > 0$ and $j > 0$; while if ∞/∞ is generated, then $i < 0$ and $j < 0$. In either case,

$$f(x) = a_i x^i + o(x^i) \qquad g(x) = b_j x^j + o(x^j)$$
$$f'(x) = ia_i x^{i-1} + o(x^{i-1}) \qquad g'(x) = jb_j x^{j-1} + o(x^{j-1}),$$

where i and j are nonzero. $\blacksquare$

Applications

One reason for the popularity of L'Hospital's Rule in calculus textbooks is the way it automates evaluating a large class of limits which might otherwise require Taylor expansion or some other special method. In the same spirit, we present five sums that are easily proved convergent by the application of Theorem 1.

(a) $\sum_{n=1}^{\infty}\left(1-n\sin\frac{1}{n}\right)$

(b) $\sum_{n=1}^{\infty} n^2\left(\cos\frac{1}{n}-1+\frac{1}{2n^2}\right)$

(c) $\sum_{n=1}^{\infty}\frac{1}{n^3\sin\frac{1}{n}}$

(d) $\sum_{n=1}^{\infty}\frac{1}{n^3\left(\ln\left(1+\frac{1}{n}\right)-1\right)}$

(e) $\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{1/n}-1\right)}$

In Application (a), the summands are $\frac{f(n)}{g(n)}$, where $f(n)=\frac{1}{n}-\sin\frac{1}{n}$ and $g(n)=\frac{1}{n}$. It is enough to study $\sum\left|\frac{f'(n)}{g'(n)}\right|$. We have

$$\left|\frac{f'(n)}{g'(n)}\right| = \left|\frac{-\frac{1}{n^2}+\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}}\right| = \left|1-\cos\frac{1}{n}\right|,$$

which the double angle formula $\cos 2\theta = 1-2\sin^2\theta$ allows us to write as $2\sin^2\frac{1}{2n}$. The inequality $|\sin\theta|\le|\theta|$ then lets us bound this by $2\left(\frac{1}{2n}\right)^2=\frac{1}{2}n^{-2}$. Putting this all together, we have

$$\sum_{n=1}^{\infty}\left|\frac{f'(n)}{g'(n)}\right| \le \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Since the series on the right hand side is convergent, by Theorem 1, $\sum_{n=1}^{\infty}\left(1-n\sin\frac{1}{n}\right)$ is absolutely convergent.

Two applications of Theorem 1 show the series of Application (b) to be absolutely convergent. In fact, an application of Theorem 1, with $f(n)=\cos\frac{1}{n}-1+\frac{1}{2n^2}$ and $g(n)=n^{-2}$, reduces Application (b) to Application (a). In each of the remaining three applications, set $f(n)=n^{-3}$. We leave working out the details of applications (b), (c), (d), and (e) as exercises for the reader.

Discrete analogues

There is a discrete version of L'Hospital's Rule called the Stolz-Cesàro Theorem. It asserts that if $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ are two sequences of real numbers with $\{a_n\}$, $\{b_n\}$ generating the indeterminate form $0/0$ at infinity with b_n strictly decreasing to 0 or with $\{a_n\}$, $\{b_n\}$ generating the indeterminate form ∞/∞ at infinity with b_n strictly increasing to ∞, and if $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ exists; then $\lim_{n\to\infty}\frac{a_n}{b_n}$ also exists and has the same value. We do not know who coined the name of this very well known theorem. The ∞/∞ case is stated and proved on pages 173–175 of Stolz's 1885 book [**4**] and also on page 54 of Cesàro's 1888 article [**2**]. It appears as Problem 70 in [**3**].

A discrete analogue of what we have done above involves investigating the conjecture that $\sum\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ converges implies that $\sum\frac{a_n}{b_n}$ converges. We will give analogues of both negative examples above and of the positive results in Theorems 1 and 2. In particular, the same distinction between the $0/0$ case and the ∞/∞ case observed in the last two sections continues to hold here as well.

EXAMPLE 3. The analogue of Example 1 requires a different construction, so we give it here.

For $2^k \le n < 2^{k+1}$, let $a_n = 4^{-k}$ and $b_n = 2^{-k} + \epsilon_n$, where ϵ_n is strictly decreasing and $0 < \epsilon_n \ll 2^{-k}$. For example, $\epsilon_n = 2^{-2^n}$ will do. Then $a_n \to 0$, $b_n \to 0$, b_n is strictly decreasing, and

$$\begin{aligned}\sum \frac{a_{n+1}-a_n}{b_{n+1}-b_n} &= \sum_{k=1}^{\infty}\left(\frac{a_{2^{k+1}-1}-a_{2^{k+1}}}{b_{2^{k+1}-1}-b_{2^{k+1}}} + \sum_{n=2^k}^{2^{k+1}-2}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}\right)\\ &= \sum_{k=1}^{\infty}\left(\frac{4^{-k}-4^{-(k+1)}}{2^{-k}-2^{-(k+1)}+\left(\epsilon_{2^{k+1}-1}-\epsilon_{2^{k+1}}\right)} + \sum_{n=2^k}^{2^{k+1}-2}\frac{0}{\epsilon_n-\epsilon_{n+1}}\right)\\ &\le \sum_{k=1}^{\infty}\frac{4^{-k}-4^{-(k+1)}}{2^{-k}-2^{-(k+1)}+0}\\ &= \sum_{k=1}^{\infty}\frac{4^{-k}}{2^{-k}}\cdot\frac{1-1/4}{1-1/2} = \frac{3}{2}\sum_{k=1}^{\infty}2^{-k} < \infty,\end{aligned}$$

so that $\sum \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ converges. On the other hand, $\sum \frac{a_n}{b_n}$ diverges, since

$$\begin{aligned}\sum_{n=2}^{2^{N+1}-1}\frac{a_n}{b_n} &= \sum_{k=1}^{N}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{4^{-k}}{2^{-k}+\epsilon_n}\right)\\ &\ge \sum_{k=1}^{N}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{4^{-k}}{2^{-k}\cdot 2}\right)\\ &= \frac{1}{2}\sum_{k=1}^{N}2^k\frac{4^{-k}}{2^{-k}} = \frac{1}{2}\sum_{k=1}^{N}1,\end{aligned}$$

which diverges as $N \to \infty$.

EXAMPLE 4. Our second counterexample is provided by Example 2.

Let $a_n = f(n)$ where $f(x) = \ln x$ and $b_n = g(n)$ where $g(x) = x\ln^2 x - 2x\ln x + 2x$. For every $n \in \mathbb{N}$, the Cauchy mean value theorem yields

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{f'(n+\theta)}{g'(n+\theta)} = \frac{1}{(n+\theta)\ln^2(n+\theta)},$$

for some $\theta = \theta(n) \in (0, 1)$, so that $\sum_n \sup_{m\ge n}\left|\frac{a_{m+1}-a_m}{b_{m+1}-b_m}\right|$ converges. But as noted in the discussion of Example 2, $\sum \frac{a_n}{b_n}$ diverges. Also $b_{n+1} - b_n = g'(n+\varphi) = \ln^2(n+\varphi)$ for some $\varphi = \varphi(n) \in (0, 1)$ for all $n \in \mathbb{N}$, so that $\{b_{n+1} - b_n\}$ is an increasing sequence.

Here is our first positive result.

THEOREM 3. *Let* $\{a_n\}$, $\{b_n\}$ *generate the indeterminate form* $0/0$ *as* $n \to \infty$. *Suppose* $\{b_n\}$ *is decreasing for all positive integers* n. *If*

$$\sum_{n=1}^{\infty} \sup_{m \geq n} \left| \frac{a_m - a_{m+1}}{b_m - b_{m+1}} \right| \tag{5}$$

converges, then $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ *converges absolutely.*

Proof. Since $\{b_n\}$ decreases strictly to 0, $\Delta b_n = b_n - b_{n+1} > 0$. Let $\Delta a_n = a_n - a_{n+1}$. Since $a_n \to 0$, for every n, $a_n = \sum_{m \geq n} \Delta a_m$. It is enough to show that for every n there exists an $m \geq n$ so that $|a_n/b_n| \leq |\Delta a_m/\Delta b_m|$. Suppose not. Then for some n and every $m \geq n$,

$$\frac{|a_n|}{b_n} = \left| \frac{a_n}{b_n} \right| > \left| \frac{\Delta a_m}{\Delta b_m} \right| = \frac{|\Delta a_m|}{\Delta b_m}$$

and so

$$|a_n| \, \Delta b_m > |\Delta a_m| \, b_n.$$

Sum these inequalities from m equal n to infinity to get a contradiction.

$$|a_n| \, b_n = |a_n| \sum_{n=m}^{\infty} \Delta b_m = \sum_{n=m}^{\infty} |a_n| \, \Delta b_m > \sum_{n=m}^{\infty} |\Delta a_m| \, b_n$$
$$\geq \left| \sum_{m=n}^{\infty} \Delta a_m \right| b_n = |a_n| \, b_n. \qquad \blacksquare$$

When $\left| \frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right|$ is decreasing in a neighborhood of infinity, we can replace the convergence of the series of suprema by the convergence of

$$\sum_{n=1}^{\infty} \left| \frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right|,$$

since

$$\left| \frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right| = \sup_{m \geq n} \left| \frac{a_m - a_{m+1}}{b_m - b_{m+1}} \right|$$

for sufficiently large n.

The following analogue of Theorem 2 also has a simple proof that is very similar to the proof of Theorem 2.

THEOREM 4. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences such that there are nonzero integers* i *and* j *and nonzero real numbers* a *and* b *and*

$$a_n = an^i + o(n^i) \qquad b_n = bn^j + o(n^j)$$
$$a_{n+1} - a_n = ian^{i-1} + o(n^{i-1}) \qquad b_{n+1} - b_n = jbn^{j-1} + o(n^{j-1}).$$

Then if $\{a_n\}$, $\{b_n\}$ *generate the indeterminate form* $0/0$ *or* ∞/∞ *at infinity and* b_n *and* $b_{n+1} - b_n$ *are not zero for all* $n \in \mathbb{N}$, *then the equivalence*

$$\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \text{ converges if and only if } \sum_{n=1}^{\infty} \frac{a_n}{b_n} \text{ converges}$$

is true.

COROLLARY. *Let f and g be functions analytic in $\mathbb{C} \setminus \{0\}$ and not having an essential singularity at infinity. If f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and $g(n)$ and $g'(n)$ are not zero for all $n \in \mathbb{N}$; then the equivalence*

$$\sum_{n=1}^{\infty} \frac{f(n+1) - f(n)}{g(n+1) - g(n)} \text{ converges if and only if } \sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \text{ converges.}$$

is true.

Proof. The first hypothesis mean that we may write

$$f(x) = \sum_{\nu=-\infty}^{i} a_\nu x^\nu \qquad g(x) = \sum_{\nu=-\infty}^{j} b_\nu x^\nu$$

where a_i and b_j are nonzero. If $0/0$ is generated, then $i > 0$ and $j > 0$; while if ∞/∞ is generated, then $i < 0$ and $j < 0$. In either case,

$$f(x) = a_i x^i + o\left(x^i\right) \qquad g(x) = b_j x^j + o\left(x^j\right)$$
$$f'(x) = i a_i x^{i-1} + o\left(x^{i-1}\right) \qquad g'(x) = j b_j x^{j-1} + o\left(x^{j-1}\right),$$

where i and j are nonzero. So there is a $\theta = \theta(n) \in (0, 1)$ such that

$$\begin{aligned} f(n+1) - f(n) &= f'(n+\theta) \\ &= i a_i (n+\theta)^{i-1} + o(n^{i-1}) \\ &= i a_i n^{i-1} + o(n^{i-1}). \end{aligned}$$

A similar calculation holds for g, so that we may finish by applying Theorem 4. ■

Acknowledgments J. Marshall Ash was supported by the Faculty and Development Program of the College of Liberal Arts and Sciences, DePaul University. Allan Berele was supported by the National Security Agency, grant H98230-08-1-0026; and by the Faculty Research Council, DePaul University.

REFERENCES

1. R. P. Boas, Counterexamples to L'Hopital's Rule, *Amer. Math. Monthly* **93** (1986) 644–645.
2. E. Cesaro, Sur la convergence des séries, *Nouvelles annales de mathématiques Series 3*, **7** (1888) 49–59.
3. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, Springer, Berlin, 1925.
4. O. Stolz, *Vorlesungen über allgemeine Arithmetik: nach den Neueren Ansichten*, Teubners, Leipzig, 1885, pp. 173–175.
5. O. Stolz, Ueber die Grenzwerthe der Quotienten, *Math. Ann.* **15** (1879) 556–559.
6. O. Stolz, *Grundzüge der Differential- und Integralrechnung*, Vol. 1., Teubners, Leipzig, 1893, pp. 72–84.
7. E. W. Weisstein, L'Hospital's Rule, from MathWorld–A Wolfram Web Resource; available at `http://mathworld.wolfram.com/LHospitalsRule.html`.

Summary Let f and g be differentiable real valued functions. Motivated by L'Hospital's Rule, we might expect the convergence of $\sum f'(n)/g'(n)$ to imply the convergence of $\sum f(n)/g(n)$ when f, g, f', g' all have limit 0 as x tends to infinity and also when all four functions have infinite limits at infinity. We find this to be true, subject to some very mild additional conditions, when the four functions have limit zero, but not necessarily to be true in the infinity case. For limits, the discrete analogue of L'Hospital's Rule is the Stolz-Cesàro Theorem. We also find a result for series that is in the spirit of the Stolz-Cesàro Theorem.

PROBLEMS

PROPOSALS

To be considered for publication, solutions should be received by July 1, 2012.

1886. *Proposed by Jodi Gubernat and Tom Beatty, Florida Gulf Coast University, Fort Myers, FL.*

For which positive integers n is the function value

$$f(n) = \sum_{k=\lfloor n/2 \rfloor}^{n} \left(1 - \frac{2k}{n}\right)^2 \binom{n}{k}$$

an integer?

1887. *Proposed by Elias Lampakis, Kiparissia, Greece.*

Given a circle $\mathcal{C}$ with center O and radius r, and a point H such that $0 < OH < r$,

(a) Show that there are an infinite number of triangles inscribed in $\mathcal{C}$ with orthocenter H.

(b) Determine the set of points belonging to the interior of all triangles inscribed in $\mathcal{C}$ with orthocenter H.

Math. Mag. **85** (2012) 61–68. doi:10.4169/math.mag.85.1.61.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1888. *Proposed by Alex Aguado, Duke University, Durham, NC.*

Let $A \subseteq X$ be a subset of a topological space, and let $N(A)$ denote the number of sets obtained from A by alternately taking closures and complements (in any order). It is well known that $N(A)$ is at most 14. However, exactly for which $r \leq 14$ is it possible to find A and X such that $N(A) = r$?

1889. *Proposed by Gary Gordon and Peter McGrath, Lafayette College, Easton, PA.*

For every positive integer k, consider the series

$$S_k = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right)$$
$$+ \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}\right) - \left(\frac{1}{3k+1} + \frac{1}{3k+2} + \cdots + \frac{1}{4k}\right) + \cdots .$$

Thus $S_1 = \log 2$ and $S_2 = (\pi + 2\log 2)/4$.

(a) Prove that S_k converges for all k.

(b) Prove that

$$S_k = \int_0^1 \frac{x^k - 1}{(x^k + 1)(x - 1)}\, dx.$$

(c) Prove that the sequence $\{S_k\}$ is monotonically increasing and divergent.

1890. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, Bronx, NY.*

Let m and n be positive integers. Prove that there exist an integer k and a prime p such that $m \equiv k^2 + p \pmod{n}$.

Quickies

Answers to the Quickies are on page 68.

Q1017. *Proposed by Allan Berele and Jeffery Bergen, Department of Mathematics, DePaul University, Chicago, IL.*

Find a monic polynomial $f(x)$ with integer coefficients such that $f(x) = 0$ has no integer solutions but $f(x) \equiv 0 \pmod{p}$ has a solution for every prime p.

Q1018. *Proposed by Finbarr Holland, School of Mathematical Sciences, University College Cork, Cork, Ireland.*

Suppose $0 < \alpha \leq 1$. Prove that

$$e^x \leq \frac{1 + (1-\alpha)x}{1 - \alpha x}$$

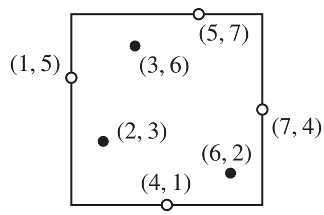
for all $x \in [0, 1)$ if and only if $\alpha \geq 1/2$.

Solutions

Permutations and their bounding squares **February 2011**

1861. *Proposed by Emeric Deutsch, Polytechnic Institute of New York University, Brooklyn, NY.*

Let $n \geq 2$ be an integer. A permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ can be represented in the plane by the set of n points $P_\sigma = \{(k, \sigma(k)) : 1 \leq k \leq n\}$. The smallest square bounding P_σ, with sides parallel to the coordinate axis, has at least 2 and at most 4 points of P_σ on its boundary. The figure below shows a permutation σ with 4 points on its bounding square. For every $m \in \{2, 3, 4\}$, determine the number of permutations σ of $\{1, 2, \ldots, n\}$ having m points of P_σ on the boundary of their bounding square.



Solution by Daniele Degiorgi, ETH Zurich, Zurich, Switzerland.

Let N_m be the number of permutations of $\{1, 2, 3, \ldots, n\}$ having m points of P on the boundary of their bounding square. All the points on the bounding square have 1 or n as x or y coordinate. There are only two points on the boundary if and only if P_σ contains both $(1, 1)$ and (n, n) or both $(1, n)$ and $(n, 1)$. Fixing the two points, none of the $(n-2)!$ permutations of $\{2, 3, \ldots, n-1\}$ will generate points on the boundary. Thus $N_2 = 2(n-2)!$.

There are four points on the boundary if P_σ does not contain any of the four points cited above $(1, 1)$, $(1, n)$, $(n, 1)$, and (n, n). To ensure this, we can select any of the $n-2$ values in $\{2, 3, \ldots, n-1\}$ for $\sigma(1)$, any of the $n-3$ values in $\{2, 3, \ldots, n-1\}$ excluding $\sigma(1)$ for $\sigma(n)$, and any of the $(n-2)!$ possibilities for the other values of of σ. Thus $N_4 = (n-2)(n-3)(n-2)!$.

Finally, $N_3 = n! - N_2 - N_4 = (4n-8)(n-2)!$.

Also solved by Michael Andreoli, Michel Bataille (France), Berry College Dead Poets Society, Jany C. Binz (Switzerland), Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Mark Bowron, Robert Calcaterra, Cal State LA Math Problem Solving Group, John Christopher, CMC 328, Con Amore Problem Group (Denmark), Calvin A. Curtindolph, Patrick Devlin, Daniel Dominik, Gregory Dresden, Dave Feil, Dmitry Fleischman, Natacha Fontes-Merz, David Getling (Germany), Arup Guha, Joshua Ide, Lucyna Kabza, Omran Kouba (Syria), Victor Y. Kutsenok, Elias Lampakis (Greece), László Lipták, Peter McPolin (Northern Ireland), David Nacin, Rituraj Nandan, Northwestern University Math Problem Solving Group, Rob Pratt, Joel Schlosberg, Thomas Q. Sibley, Skidmore College Problem Group, John H. Smith, Philip Straffin, Marian Tetiva (Romania), R. S. Tiberio, Texas State University Problem Solving Group, Dennis Walsh, Michael Woltermann, and the proposer.

Majorization: sum implies product **February 2011**

1862. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let n be a positive integer. Suppose that the nonnegative real numbers $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ satisfy that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$ for all $1 \leq k \leq n$. Prove that $\prod_{i=1}^{k} a_i \geq \prod_{i=1}^{k} b_i$ for all $1 \leq k \leq n$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The desired inequalities are trivially true if $a_1 = 0$, because $a_1 = 0$ implies $b_1 = 0$, so let us suppose that $a_1 > 0$.

For every nonnegative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, we have that

$$\sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{k} b_i \right) \le \sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{k} a_i \right).$$

This is equivalent to

$$\sum_{i=1}^{n} \Lambda_i b_i \le \sum_{i=1}^{n} \Lambda_i a_i,$$

where $\Lambda_i = \lambda_i + \lambda_{i+1} + \cdots + \lambda_n$.

Now consider $k \in \{1, 2, \ldots, n\}$ and suppose that $\Lambda_k > 0$. By the previous inequality and the Arithmetic Mean–Geometric Mean Inequality, it follows that

$$\left(\prod_{i=1}^{k} b_i \right)^{1/k} = \frac{1}{\sqrt[k]{\Lambda_1 \cdots \Lambda_k}} \left(\prod_{i=1}^{k} \Lambda_i b_i \right)^{1/k} \le \frac{1}{\sqrt[k]{\Lambda_1 \cdots \Lambda_k}} \cdot \frac{1}{k} \sum_{i=1}^{k} \Lambda_i b_i$$
$$\le \frac{1}{\sqrt[k]{\Lambda_1 \cdots \Lambda_k}} \cdot \frac{1}{k} \sum_{i=1}^{k} \Lambda_i a_i.$$

Choose the λ_i as follows:

$$\lambda_i = \begin{cases} 0 & \text{if } k < i \le n, \\ 1/a_k & \text{if } i = k, \\ 1/a_i - 1/a_{i+1} & \text{if } 1 \le i < k. \end{cases}$$

Note that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nonnegative because $a_1 \le a_2 \le \cdots \le a_k$ and moreover $\Lambda_i = 1/a_i$ for $1 \le i \le k$, and $\Lambda_i = 0$ for $k < i \le n$. Thus

$$\left(\prod_{i=1}^{k} b_i \right)^{1/k} \le \frac{1}{\sqrt[k]{\frac{1}{a_1} \cdots \frac{1}{a_k}}} \cdot \frac{1}{k} \sum_{i=1}^{k} 1 = \left(\prod_{i=1}^{k} a_i \right)^{1/k},$$

which is the desired inequality for k.

Editor's Note. Several solvers pointed out that the inequality is a special case of Karamata's Inequality and involves the idea of majorization. A non-increasing n-tuple $a = (a_1, a_2, \ldots, a_n)$ majorizes another non-increasing n-tuple $b = (b_1, b_2, \ldots, b_n)$ if $\sum_{i=1}^{k} a_i \ge \sum_{i=1}^{k} b_i$ for $k = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. One form of Karamata's Inequality states that if $\phi(t)$ is a continuous, convex function, and a majorizes b, then

$$\sum_{i=1}^{n} \phi(a_i) \ge \sum_{i=1}^{n} \phi(b_i).$$

By choosing $\phi(t) = \log(t)$ and rearranging the sequences, the solution is a special case of Karamata's Inequality.

Also solved by Elton Bojaxhiu (Germany) and Enkel Hysnelaj(Australia); Robert Calcaterra; Marian Dinča (Romania); Robert L. Doucette; John N. Fitch; Vikram Govindan; Lixing Han; Peter W. Lindstrom; László Lipták; Larry A. Lucas, Faryal Bokhari, and Touissant Towa; Peter McPolin (Northern Ireland); Paolo Perfetti (Italy); Joel Schlosberg; Marian Tetiva (Romania); and the proposer.

Moments, $||f'||_\infty$, and Newton–Cotes formulae **February 2011**

1863. *Proposed by Duong Viet Thong, Department of Economics and Mathematics, National Economics University, Hanoi, Vietnam.*

Let f be a continuously differentiable function on $[a, b]$ such that $\int_a^b f(x)\,dx = 0$. Prove that

$$\left|\int_a^b xf(x)\,dx\right| \le \frac{(b-a)^3}{12}\max\{|f'(x)| : x \in [a, b]\}.$$

Solution by Sanghun Song, Seoul Science High School, Jongro-ku, Seoul, Korea.

Let $M = \max\{|f'(x)| : x \in [a, b]\}$, c the midpoint of the interval $[a, b]$, and ℓ its semi-length, i.e., $c = \frac{1}{2}(a+b)$ and $\ell = \frac{1}{2}(b-a)$. Since $\int_a^b f(x)dx = 0$, we have that

$$\int_a^b xf(x)\,dx = \int_a^b (x-c)f(x)\,dx = \int_a^c (x-c)f(x)\,dx + \int_c^b (x-c)f(x)\,dx.$$

Using the change of variable $t = |x-c|$ gives

$$\begin{aligned}\int_a^b xf(x)\,dx &= -\int_0^\ell tf(c-t)\,dt + \int_0^\ell tf(c+t)\,dt\\ &= \int_0^\ell t(f(c+t) - f(c-t))\,dt.\end{aligned}$$

Now, the Mean Value Theorem implies that for every $t \in [0, \ell]$ there is $\xi_t \in (c-t, c+t)$ such that $|f(c+t) - f(c-t)| = |(c+t) - (c-t)||f'(\xi_t)| \le 2tM$. Hence, putting these two facts together we obtain

$$\left|\int_a^b xf(x)\,dx\right| \le \int_0^\ell t|f(c+t) - f(c-t)|\,dt \le 2M\int_0^\ell t^2\,dt = \frac{2\ell^3}{3}M,$$

which is exactly the required inequality.

Editor's Note. Many solutions were based, more or less, on variations of the above argument. Some solvers used a different idea, starting with the twice differentiable function $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$, they observed that $\int_a^b xf(x)\,dx = -\int_a^b F(x)\,dx$, and then used the *l*ocal error for the trapezoid rule also known as one of the Newton–Cotes formulae,

$$\int_a^b F(x)\,dx = (F(a) + F(b))\ell - \frac{2\ell^3}{3}F''(c) \text{ for some } c \in (a, b).$$

Another idea used by R. Calcaterra and D. Dominik was to show that the functional $f \to \int_a^b xf(x)\,dx$ attains its extreme values on the convex set $\{f \in C^1[a, b] : \int_a^b f(x)\,dx = 0, \text{ and } ||f'||_\infty \le 1\}$ only for the linear cases, i.e., $f(x) = \pm(x-c)$ with $c \in [a, b]$. Finally it was brought to our attention by P. Perfetti that one can use

Problem E2155 in *American Mathematical Monthly* (December 1969, pp. 1142–1143) to prove the following generalization:

$$\left|\int_a^b x^n f(x)\,dx\right| \le \frac{(n!)^2(b-a)^{2n+1}}{(2n!)(2n+1)!}M,$$

for all f such that $\int_a^b x^k f(x)\,dx = 0$ for $k = 0, 1, 2, \ldots, n-1$.

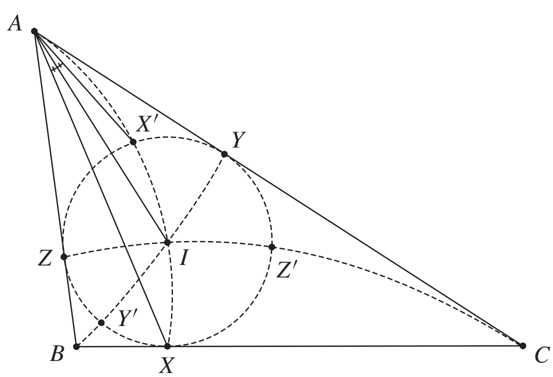
Other connections with the proposed problem can be found in Ch. XV of the book *Inequalities involving functions and their integrals and derivatives* written by D. S. Mitrinović, J. E. Pečaric, and A. M. Fink and printed by Kluwer Academic Publishers, 1991.

Also solved by Armstrong Problem Solvers; Cody M. Allen and William R. Green; Michel Bataille (France); Dionne Bailey, Elsie Campbell, and Charles Diminnie; Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia); Michael W. Botsko; Robert Calcaterra; Hongwei Chen; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal (Spain); Daniel Dominik; Robert L. Doucette; Josh Eyler; Omran Kouba; Charles Lindsay; Peter W. Lindstrom; Rick Mabry; Raymond Mortini (France); Scott Pauley, Andrew Welter, and Natalya Weir; Paolo Perfetti (Italy); Ángel Plaza (Spain); Rob Pratt; Henry Ricardo; Joel Schlosberg; Allen Stenger; Richard Stephens; Xiao Tingben (China); Texas State University Problem Solving Group; Haohao Wang, Jerzy Wojdyło, and Yanping Xia; Luyuan Yu (China); and the proposer. There were three incorrect submissions.

Isogonal conjugate concurrent cevians **February 2011**

1864. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ.*

Let ABC be a scalene triangle, I its incenter, and X, Y, and Z the tangency points of its incircle $\mathcal{C}$ with the sides BC, CA, and AB, respectively. Denote by $X' \neq X$, $Y' \neq Y$, and $Z' \neq Z$ the intersections of $\mathcal{C}$ with the circumcircles of triangles AIX, BIY, and CIZ, respectively. Prove that the lines AX', BY', and CZ' are concurrent.



Solution by Elton Bojaxhiu, Kriftel, Germany, and Enkel Hysnelaj, University of Technology, Sydney, Australia.

Because $IX = IX'$ and A, X', I, and X are cyclic, it follows that $\angle XAI = \angle IAX'$. And since AI is the bisector of $\angle BAC$, we conclude that the lines AX and AX' are symmetrical with respect to this bisector line. Similarly the lines BY' and BY are symmetrical with respect to the line BI, and the lines CZ' and CZ are symmetrical with respect to the line CI.

It is a well-known direct application of Ceva's Theorem that AX, BY, and CZ are concurrent. Since the lines AX', BY', and CZ' are the symmetrical of AX, BY, and CZ with respect to the bisectors AI, BI, and CI, respectively, this implies that the lines AX', BY', and CZ' are concurrent. This fact also follows directly from Ceva's Theorem.

Editor's Note. The intersection of the lines AX, BY, and CZ is called the Gergonne point of $\triangle ABC$. Joel Schlosberg points out that the intersection of the lines AX', BY', and CZ' is the isogonal conjugate of the Gergonne point, and is referred to as the Insimilicenter of the Circumcircle and the Incircle (point X(55)) in Clark Kimberling's Encyclopedia of Triangle Centers available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html. Omran Kouba notes that the required point of concurrency lies on the Euler line of $\triangle XYZ$. Finally, the details of the last part of the proof can be obtained from the trigonometric version of Ceva's Theorem, i.e., AX', BY', and CZ' are concurrent if and only if

$$\frac{\sin\angle BAX' \cdot \sin\angle CBY' \cdot \sin\angle ACZ'}{\sin\angle X'AC \cdot \sin\angle Y'BA \cdot \sin\angle Z'CB} = 1.$$

Also solved by Michel Bataille (France), John G. Heuver (Canada), Omran Kouba (Syria), Joel Schlosberg, Ercole Suppa (Italy), and the proposer.

Reducing representations of a ring **February 2011**

1865. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, Bronx, NY.*

In the solution to Problem 1790 (this *Magazine* **82** (2009) 67–68), it was proved that if R is a ring such that for each element $x \in R$,

$$x + x^2 + x^3 + x^4 = x^{11} + x^{12} + x^{13} + x^{28},$$

then for each element $x \in R$, $x = x^{127}$. Under the same hypothesis, prove that for each element $x \in R$, $6x = 0$ and $x = x^7$.

Solution by Robert Calcaterra, University of Wisconsin–Platteville, Platteville, WI.

Let $f(x) = x^{28} + x^{13} + x^{12} + x^{11} - x^4 - x^3 - x^2 - x$. Fix $c \in R \setminus \{0\}$. Since the zero element obviously satisfies the required conclusions, the proof will be complete if we show that $6c = 0$ and $c^7 = c$.

Let T be the set of all polynomial expressions in c with integer coefficients and constant term 0, i.e., $T = c\mathbb{Z}[c]$. Observe that T is a commutative ring. Moreover, if $e = c^{27} + c^{12} + c^{11} + c^{10} - c^3 - c^2 - c$, then $ce = c$ and so the ring T has unity e. Let $g(x) \in x\mathbb{Z}[x]$. If we divide $g(x)$ by $f(x)$ using the division algorithm, then the reminder $r(x)$ will have the property that $g(c) = r(c)$ because $f(c) = 0$. Consequently, every element of T is an integer combination of $\{c, c^2, c^3, \ldots, c^{27}\}$.

Let n be the order of e in the additive group of T. Assume n is infinite. Then $S := \{ke : k \in \mathbb{Z}\}$ is a subring of T that is isomorphic to the ring of integers. Since S can be embedded in a field, the number of roots of f in S cannot exceed 28. This contradiction forces us to conclude that n is finite. This conclusion further implies that the order of every element in the group $(T, +)$ is a divisor of n. Therefore, the number of distinct elements of T is at most n^{27}. In particular T is a finite ring.

If $b \in R$ and $b^2 = 0$, then $f(b) = 0$ implies that

$$b = b^2(b^{26} + b^{11} + b^{10} + b^9 - b^2 - b) - b^2 = 0.$$

Hence, zero is the only element of R whose square is zero. Therefore, by the Wedderburn–Artin Theorem, T is isomorphic to a finite direct sum of full matrix rings over division rings. Moreover, since T is commutative, these matrices must be 1 by 1 and the components must come from a field. In other words, T must be isomorphic to a direct sum of finite fields.

Let $\mathbb{F}$ be a subfield of T that has q elements (q is a power of a prime). Lagrange's Theorem implies that every element of $\mathbb{F}$ is a root of $x^q - x$. All the roots of this polynomial are simple and so $x^q - x$ must be a divisor of $f(x)$ in the ring $\mathbb{F}[x]$. Therefore, we may use routine computation to show that q is 2, 3, or 4. Hence the characteristic of $\mathbb{F}$ is either 2 or 3 and thus $6x = 0$ for all $x \in \mathbb{F}$. In addition, the order of an element of $(\mathbb{F} \setminus \{0\}, \cdot)$ must be either 1, 2, or 3 and so $x^7 = x$ for all $x \in \mathbb{F}$. Since T is a direct sum of such fields, it follows that $6c = 0$ and $c^7 = c$. This completes the proof.

Note. If $GF(q)$ denotes the field with q elements, then the ring $GF(3) \oplus GF(4)$ meets the hypothesis of the problem and the equation $x^k = x$ is not satisfied by at least one element of this ring when $k < 7$ is a positive integer. Hence 7 is the least positive integer for which the conclusion of this problem is true.

Also solved by John Riegsecker and the proposer.

Answers

Solutions to the Quickies from page 62.

A1017. In $\mathbb{Z}_p^*$, the multiplicative group of non-zero elements of $\mathbb{Z}_p$, the squares form a subgroup of index 2 and so the product of any two non-squares is a square. Hence, at least one of -1, 2 or -2 will be a square. Thus one polynomial that satisfies the requirements is

$$f(x) = (x^2 + 1)(x^2 - 2)(x^2 + 2).$$

A1018. If the inequality is true, then so is the following statement:

$$\frac{e^x - x - 1}{x(e^x - 1)} \leq \alpha$$

for all $x \in (0, 1)$. But applying L'Hôpital's rule twice gives

$$\alpha \geq \lim_{x \to 0^+} \frac{e^x - x - 1}{x(e^x - 1)} = \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + e^x - 1} = \lim_{x \to 0^+} \frac{e^x}{e^x(x + 2)} = \frac{1}{2}.$$

Thus, the condition on α is necessary. Conversely, if $\alpha \geq 1/2$, then $1/n! \leq 1/2^{n-1} \leq \alpha^{n-1}$ for every positive integer n, and so, if $0 \leq x < 1$, then

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \leq 1 + \sum_{n=1}^{\infty} \alpha^{n-1} x^n = \frac{1 + (1 - \alpha)x}{1 - \alpha x},$$

as required, with equality if and only if $x = 0$.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Dash, Mike, Past imperfect: The woman who bested the men at math, http://blogs.smithsonianmag.com/history/2011/10/the-woman-who-bested-the-men-at-math/.

Philippa Fawcett placed first—"above the [male] senior Wrangler"—in the Cambridge Mathematical Tripos in 1890, the only time such an "upset" occurred. At that time women's results in the exams were marked, ranked, and read separately, and women were ineligible for Cambridge degrees. The exam comprised 12 papers of 16 problems each, and 5.5 hours of work every day for eight days; those in contention for high places sat for three more days and 63 more problems. Ms. Fawcett received a degree instead from the University of London, was a lecturer at Newnham College at Cambridge for 10 years, set up schools in South Africa, and made a career as an educational administrator for the London County Council. Women were finally allowed to take Cambridge University degrees in 1948, one month before her death.

MathDL: The MAA Mathematical Sciences Digital Library, http://mathdl.maa.org/mathDL/.

Math in the Media, http://www.ams.org/news/math-in-the-media/mmarc-index.

I gave an invited talk at my alma mater on "How to Keep Up with Mathematics," and for a few years after I gave revised versions to senior mathematics majors, with advice about general publications in science, mathematics indexes, books and journals, and Web sources. The talk is out of date! Since then, many more resources have become "available." What does "available" mean? A colleague makes wonderful resources for his courses, and for department members, "available": The materials are somewhere at his Website, at an address that he presumes you know but you are unlikely to remember. *Web "publication" is not distribution of information.* In the Pre-Digital Age, to read a journal issue you could go to the library; alternatively, a subscription brought it to your mailbox. Disregarding cost, what would be your preferred method of access? Today, the MAA makes "available" its Mathematical Sciences Digital Library (MathDL), a portal that combines notices of media coverage of math, reviews, On This Day…, and *Loci* (a new bag for the contents of *Convergence* plus the former *J. Online Mathematics and Its Applications* and Digital Classroom Resources). Similarly, the American Mathematical Society (AMS) makes "available" Math in the Media, with its own summaries of media coverage of math, a monthly digest of mathematical news, book reviews (none from this column—shame on them!), and more. Yes, it is all "available." But to find out what is there, you have to do the contemporary equivalent of walking to the library—go to the Websites—regularly. I don't do that; do you? The AMS offers the capability to *send* notice to others of what you find, via Facebook and Twitter (and more services), but not to *receive* automatic notice of contents that might interest you. The MAA is more friendly in that regard, offering RSS feeds to your email inbox for MathDL items (but why not for the MAA Columns? Yes, notice of those is posted in the monthly email MAA Math Alert). You may ask, do I want still more stuff in my inbox? Really good stuff from the AMS and the MAA, you bet!

Math. Mag. **85** (2012) 69–70. doi:10.4169/math.mag.85.1.69. © Mathematical Association of America

Bower, Bruce, Skateboarders rock at physics, *Science News* 180 (12) (3 December 2011) 10.

Denike, Bob, Cycloid ramp theory, *Thrasher Magazine* (September 1985), 38–39.

Performa 11: Raphaël Zarka, Free ride. http://11.performa-arts.org/event/raphael-zarka-performa-project.

Skateboarders realize that a steeper slope at the start of a descent can result in a faster time, but most college students do not. Although skateboard courses tend toward half-pipes with flat bottoms, plans for a cycloidal ramp were published already in 1985, concluding "we'll have to wait for someone to make a cycloid ramp to see if it really works." Well, finally!: In November at Performa 11 in New York, artist Raphaël Zarka was to produce the "world's first cycloid skateboard ramp." Unfortunately, by the time students reach the brachistochrone in calculus (if their course offers such applications), they have left skateboards behind; but the same principles apply to snowboarding, in which girls take more interest than for almost-all-boys skateboarding.

Freedman, David H., A formula for economic calamity, *Scientific American* 305 (5) (November 2011) 77–79.

Ehrenberg, Rachel, Beware the long tail, *Science News* 180 (10) (5 November 2011) 22–25.

Rogalski, Marc, Mathematics and finance: An ethical malaise, *Mathematical Intelligencer* 33 (2) (Summer 2010) 6–8. Ekeland, I., Response to Rogalski, *ibid.* 9–10.

Korman, Jonathan, Finance and mathematics: A lack of debate, *Mathematical Intelligencer* 33 (2) (Summer 2011) 4–6.

Carnevale, Anthony P., Nicole Smith, and Michelle Melton, STEM: Science, technology, engineering and mathematics, http://www9.georgetown.edu/grad/gppi/hpi/cew/pdfs/stem-complete.pdf.

Freedman asserts, "Wall Street is betting our future on flimsy science." The science is mathematics. Do we deserve the blame? The models in question omitted consideration of liquidity of portfolios of bundled mortgages and assumed that the risks run by different portfolio holders were independent. "The only real option is not to trust the models,. . . but 'the people in control keep making lots of money using them. . . Now they're trusting them again.'"

Ehrenberg notes that extreme events are much more common than expected under the common assumption of an underlying normal distribution (e.g., for the Black-Scholes pricing results). Power laws have heavier tails, and the "rare" events (which are less-rare under a power-law distribution) can be precipitated by leveraging of investments; but sometimes crashes are "in a class of their own," beyond the ability of power law distributions to justify.

Korman echoes Rogalski about the need for protection from the financial industry, pointing to mathematicians. Carnevale et al. show that the U.S. produces enough STEM graduates—but too few stay in STEM careers. Korman cites Toronto mathematics Ph.D.'s going into the financial industry and suggests that we should be concerned that "our students. . . work for institutions known for looting the public"; "[Mathematics departments] are complicit in. . . training students in research areas where nobody is hiring, leaving them without room for maneuver when it comes to job-hunting. In effect, departments are handing over many of their students to the banks.. . . Departments. . . should encourage students to think about their role in society."

Simoson, Andrew J., *Voltaire's Riddle: Micromégas and the Measure of All Things*, MAA, 2010; xvii + 377 pp, $58.95 (member price: $47.95). ISBN 978-0-88385-345-0.

I reviewed author Simoson's previous extraordinary book, *Hesiod's Anvil* in the December 2007 issue of THIS MAGAZINE (I notice now that I misspelled his name, but no one complained); it dealt with gravitation, falling, and trajectories. Like it, *Voltaire's Riddle* explores mathematics in an unusual and fascinating fashion: as introduced in or occasioned by literature, in this case a fictional tale by Voltaire about measurement of the Earth. Vignettes introduce chapters on ellipses, hypocycloids, π, arclength, torque, trajectories, and more; there are numerous figures and portraits, plus exercises with selected answers and "comments." Voltaire advised, "Avoid pleasantry in mathematics." But as Simoson remarks, "This entire volume is a counterexample. . . ," and he is right indeed. The book builds a valuable bridge between the Two Cultures; but since the reader needs vector calculus and linear algebra, it's a bridge in only one direction.

NEWS AND LETTERS

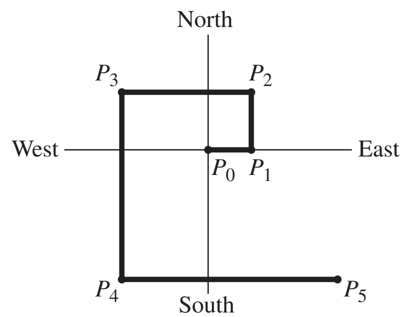
72nd Annual William Lowell Putnam Mathematical Competition

Editor's Note: Additional solutions will be printed in the *Monthly* later in the year.

PROBLEMS

A1. Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0, 0), P_1, \ldots, P_n$ such that $n \geq 2$ and:

- The directed line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.
- The lengths of these line segments are positive and strictly increasing.



How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n, of any growing spiral?

A2. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1{=}b_1{=}1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S.

A3. Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \displaystyle\int_0^{\pi/2} x^r \sin x \, dx}{\displaystyle\int_0^{\pi/2} x^r \cos x \, dx} = L.$$

A4. For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

A5. Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u, u) = 0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u, v) \in \mathbb{R}^2$, the vector $\nabla F(u, v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u), -g(v)\rangle$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \ldots, x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.$$

Math. Mag. **85** (2012) 71–78. doi:10.4169/math.mag.85.1.71.

A6. Let G be an abelian group with n elements, and let

$$\{g_1 = e, g_2, \ldots, g_k\} \subsetneqq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, \ldots, g_k$ with equal probability, is rolled m times, and the selected elements are multiplied to produce an element $g \in G$.

Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m\to\infty} \frac{1}{b^{2m}} \sum_{x\in G} \left(\mathrm{Prob}(g = x) - \frac{1}{n}\right)^2$$

is positive and finite.

B1. Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

B2. Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S ?

B3. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

B4. In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

B5. Let $a_1, a_2, \ldots$ be real numbers. Suppose that there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \Big(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2}\Big)^2 dx \le An.$$

Prove that there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} \big(1 + (a_i - a_j)^2\big) \ge Bn^3.$$

B6. Let p be an odd prime. Show that for at least $(p + 1)/2$ values of n in $\{0, 1, 2, \ldots, p - 1\}$,

$$\sum_{k=0}^{p-1} k!\, n^k \qquad \text{is not divisible by } p.$$

SOLUTIONS

Solution to A1. If $1 \le m < n$, then the point (m, n) is reached in two steps with the growing spiral $(0, 0)$, $(m, 0)$, (m, n). These are the only points that can be reached in two steps.

If $4 \le n \le m$, then the point (m, n) is reached in six steps with the growing spiral

$$(0, 0),\quad (1, 0),\quad (1, 2),\quad (-2, 2),\quad (-2, -3 - m + n),\quad (m, -3 - m + n),\quad (m, n).$$

On the other hand, because the lengths of the line segments are strictly increasing and segment $P_{k+1}P_{k+2}$ is always in the direction opposite $P_{k-1}P_k$, for any growing spiral the only points in the first quadrant are P_2, P_6, $P_{10}, \ldots$, and the change in each coordinate from one of these points to the next is at least 2. Thus the points listed above are the only points in the first quadrant that can be reached by growing spirals of any length.

The remaining, unreachable, lattice points in our square region are (m, n) with …

$m = 0,$	any n	(2012 points)
$m = 1,$	$n = 0, 1$	(2 points)
$m = 2,$	$n = 0, 1, 2$	(3 points)
$3 \le m \le 2011,$	$n = 0, 1, 2, 3$	(4 points for each of 2009 m-values)

… making $2012 + 2 + 3 + 4(2009) = 10053$ unreachable points in all.

Solution to A2. For $n = 2, 3, \ldots$ we have

$$\frac{1}{a_1 \cdots a_n} = \frac{b_{n-1}a_n - b_n}{2a_1 \cdots a_n} = \frac{1}{2}\left(\frac{b_{n-1}}{a_1 \cdots a_{n-1}} - \frac{b_n}{a_1 \cdots a_n}\right).$$

Therefore,

$$\sum_{n=2}^{k} \frac{1}{a_1 \cdots a_n} = \frac{1}{2}\sum_{n=2}^{k}\left(\frac{b_{n-1}}{a_1 \cdots a_{n-1}} - \frac{b_n}{a_1 \cdots a_n}\right) = \frac{1}{2}\left(\frac{b_1}{a_1} - \frac{b_k}{a_1 \cdots a_k}\right).$$

We claim that $\dfrac{b_k}{a_1 \cdots a_k} \to 0$ as $k \to \infty$. In fact, using that $a_n = (b_n + 2)/b_{n-1}$ for $n \ge 2$ and $a_1 = b_1 = 1$, we have

$$\frac{b_k}{a_1 \cdots a_k} = \frac{b_1 \cdots b_{k-1}b_k}{(b_2 + 2) \cdots (b_k + 2)} = \prod_{n=2}^{k} \frac{b_n}{b_n + 2}.$$

Because we are assuming that $0 < b_n \le M$ for some M, the last product is positive and bounded by $(M/(M+2))^{k-1}$, which shows that it approaches 0 as $k \to \infty$. It follows that S is convergent and that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n} = \frac{1}{a_1} + \frac{1}{2}\frac{b_1}{a_1} = \frac{3}{2}.$$

Solution to A3. The statement is true if and only if $c = -1$ and $L = 2/\pi$.

Integrating by parts, we get

$$\int_0^{\pi/2} x^r \sin x \, dx = \frac{(\pi/2)^{r+1}}{r+1} - \int_0^{\pi/2} \frac{1}{r+1} x^{r+1} \cos x \, dx.$$

We have

$$0 < \int_0^{\pi/2} \frac{1}{r+1} x^{r+1} \cos x \, dx < \int_0^{\pi/2} \frac{1}{r+1} x^{r+1} \, dx = \frac{1}{(r+1)(r+2)} (\pi/2)^{r+2}.$$

Thus,

$$\lim_{r \to \infty} \frac{\int_0^{\pi/2} x^r \sin x \, dx}{\frac{(\pi/2)^{r+1}}{r+1}} = 1.$$

On the other hand, integrating the original denominator by parts, we get

$$\int_0^{\pi/2} x^r \cos x \, dx = \frac{1}{r+1} \int_0^{\pi/2} x^{r+1} \sin x \, dx$$

and hence, using the previous limit,

$$\lim_{r\to\infty} \frac{\int_0^{\pi/2} x^r \cos x\, dx}{\frac{(\pi/2)^{r+2}}{(r+1)(r+2)}} = 1.$$

Taking the ratio of the two limits proves our claim.

Solution to A4 (based on a student paper). There is such a matrix if and only if n is odd.

If n is odd, we can take the matrix whose diagonal entries are all 0 and whose other entries are all 1.

Suppose that n is even and that A is a matrix with the desired properties. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the rows of A, and let $\mathbf{r} = (1, \dots, 1)$ be the n-dimensional row vector of ones. Note that

$$\mathbf{v}_j \cdot \mathbf{r} \equiv \mathbf{v}_j \cdot \mathbf{v}_j \equiv 0 \pmod 2.$$

Therefore, replacing any particular row $\mathbf{v}_i$ by $\mathbf{v}_i + \mathbf{r}$ does not affect the parity of the dot product of that row with any other row. It also doesn't affect the parity of the dot product of the row with itself, because

$$\begin{aligned}(\mathbf{v}_i + \mathbf{r}) \cdot (\mathbf{v}_i + \mathbf{r}) &\equiv \mathbf{v}_i \cdot \mathbf{v}_i + \mathbf{r} \cdot \mathbf{r} \\ &= \mathbf{v}_i \cdot \mathbf{v}_i + n \\ &\equiv \mathbf{v}_i \cdot \mathbf{v}_i \pmod 2.\end{aligned}$$

It follows that we may assume that the first column of A consists entirely of ones. But then if we remove the first column, that will switch the parity of all the dot products and so we will have n $(n-1)$-dimensional row vectors that are orthonormal over the field of integers mod 2, a contradiction.

Solution to A5. From the condition that $x^2 g(x) \le 1$, we know that the improper integral $\int_{-\infty}^{\infty} g(x)\, dx$ converges. Let $\Delta = \int_{-\infty}^{\infty} g(x)\, dx$.

Now for all $u, v \in \mathbb{R}$, define $\Phi(u, v) = \int_v^u g(x)\, dx$. Then $\nabla\Phi(u, v) = \langle g(u), -g(v)\rangle$ for all u, v. Since the gradients of Φ and F are parallel, these two functions have the same level curves. (More precisely: The level sets of Φ are smooth curves normal at each point to $\nabla\Phi$. The gradient of F, whether it is zero or is parallel to $\nabla\Phi$, has a zero component along one of these level curves. Therefore F is constant on each level curve of Φ.) It follows that

$$F(u, v) = f(\Phi(u, v))$$

for some continuously differentiable function $f : (-\Delta, \Delta) \to \mathbb{R}$ with $f(0) = 0$.

We note that the number

$$\Delta' = \max_{t\in[0,\Delta/2]} \left| \frac{f(t)}{t} \right|$$

is finite. (Here the expression $f(t)/t$ is to be interpreted as $f'(0)$ when $t = 0$.)

Now, given any $n + 1$ numbers $x_1, \dots, x_{n+1} \in \mathbb{R}$, relabel them if necessary so that $x_1 \le x_2 \le \cdots \le x_{n+1}$. Then we have

$$\begin{aligned}&\Phi(x_2, x_1) + \Phi(x_3, x_2) + \cdots + \Phi(x_{n+1}, x_n) \\ &\quad = \int_{x_1}^{x_2} g(x)\, dx + \int_{x_2}^{x_3} g(x)\, dx + \cdots + \int_{x_n}^{x_{n+1}} g(x)\, dx \\ &\quad < \int_{-\infty}^{\infty} g(x)\, dx = \Delta.\end{aligned}$$

It follows that at least one of the summands on the left, say the one involving x_i and x_{i+1}, is less than Δ/n. Then

$$|F(x_{i+1}, x_i)| = |f(\Phi(x_{i+1}, x_i))| \leq \max_{t \in [0, \Delta/n]} |f(t)|$$
$$\leq \frac{\Delta}{n} \max_{t \in [0, \Delta/n]} \left| \frac{f(t)}{t} \right| \leq \frac{\Delta\Delta'}{n}.$$

Therefore, we may take $C = \Delta\Delta'$.

Solution to A6. Consider the vector of probabilities

$$\mathbf{p} = [\text{Prob}(g = x)]^T$$

indexed by the group element x. Multiplying by a particular group element acts as a permutation matrix on **p**. Because a permutation matrix is orthogonal, it is diagonalizable by a unitary matrix, and its eigenvalues have complex absolute value 1.

A roll of the die is represented by a doubly stochastic matrix P (that is, a matrix in which all entries are nonnegative and all rows and columns sum to 1) which is the mean of k such permutation matrices. Because G is abelian, the permutation matrices commute with each other, so they are simultaneously diagonalizable by a unitary matrix. That is, they have a common orthogonal basis of eigenvectors in $\mathbb{C}^n$, which are also eigenvectors for P. Each eigenvalue of P is the mean of k complex numbers of absolute value 1, one of which (corresponding to $g_1 = e$) equals 1. It follows that all eigenvalues of P except 1 have absolute value (strictly) less than 1. Moreover, any eigenvector of P for $\lambda = 1$ is also an eigenvector for $\lambda = 1$ of all k the permutation matrices; because $\{g_1, g_2, \ldots, g_k\}$ generates G, we see that the eigenspace for the eigenvalue 1 is spanned by the probability vector $[1/n, \ldots, 1/n]^T$. Thus by orthogonality, eigenvectors for the other eigenvalues have entries that sum to 0.

Because the sum of the entries of $[1, 0, \ldots, 0]^T$ is 1, this vector can be written as follows as a sum of orthogonal eigenvectors:

$$[1, 0, \ldots, 0]^T = [1/n, \ldots, 1/n]^T + \mathbf{v}_2 + \cdots + \mathbf{v}_j.$$

Let $P\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Then not all the λ_i can be zero, because the generating set is not all of G and therefore $P[1, 0, \ldots, 0]^T \neq [1/n, \ldots, 1/n]^T$. We have

$$P^m[1, 0, \ldots, 0]^T = [1/n, \ldots, 1/n]^T + \lambda_2^m \mathbf{v}_2 + \cdots + \lambda_j^m \mathbf{v}_j,$$

and the components of $P^m[1, 0, \ldots, 0]^T$ are the various probabilities $\text{Prob}(g = x)$. Therefore, by the Pythagorean theorem,

$$\sum_{x \in G} \left(\text{Prob}(g = x) - \frac{1}{n} \right)^2 = \| P^m[1, 0, \ldots, 0]^T - [1/n, \ldots, 1/n]^T \|^2$$
$$= |\lambda_2|^{2m} \|\mathbf{v}_2\|^2 + \cdots + |\lambda_j|^{2m} \|\mathbf{v}_j\|^2.$$

This shows that the statement will be true for $b = \max\{|\lambda_i|\}$. (The limit will be the sum of the squares of the lengths of the corresponding eigenvectors.)

Solution to B1 (based on a student paper)**.** Note that for $0 < x < 3$ we have

$$1 + \frac{x}{3} < \sqrt{1 + x} < 1 + \frac{x}{2}.$$

Subtracting 1 and letting $x = \dfrac{p}{q^2k^2}$, we see that

$$\frac{ph}{3qk} < h\sqrt{q^2k^2 + p} - k\sqrt{q^2h^2} < \frac{ph}{2qk}.$$

Thus it is enough if we can find positive integers p and q such that $\frac{p}{q^2k^2} < 3$ and

$$\frac{3k\varepsilon}{h} < \frac{p}{q} < \frac{4k\varepsilon}{h}. \tag{1}$$

Note that doubling both p and q leaves $\frac{p}{q}$ unchanged, but halves $\frac{p}{q^2k^2}$. Therefore, it is enough to satisfy (1), but this can be done by the density of $\mathbb{Q}$ in $\mathbb{R}$.

Solution to B2 (based on a student paper). The primes that appear in seven or more elements of S are 2 and 5.

Given prime numbers p, q, r, any rational number x satisfying $px^2 + qx + r = 0$ is certainly negative, so by the rational root theorem, the only possibilities for x are -1, $-r$, $-1/p$, and $-r/p$. Candidates $x = -1$ and $x = -r/p$ are solutions if and only if $q = p + r$, so in this case one of p and r is 2 and the other forms a pair of twin primes with q. Candidates $x = -r$ and $x = -1/p$ are solutions if and only if $pr = q - 1$, so again one of p and r must be 2, while the other forms a pair of Germain primes with q (that is, q and $(q-1)/2$ are both prime). Because for any integer a one of $a, a+2, a+4$ is divisible by 3, any prime greater than 5 cannot be part of more than one pair of twin primes, so it can appear in at most two elements of S in the first case (once for each order of p and r). It can also appear in at most four elements of S in the second case, for a total of at most six elements of S. Similarly, because 1 is not prime, 3 appears in just four elements of S. Meanwhile, the primes 2 and 5 both appear in the seven elements $(2, 5, 2)$, $(2, 5, 3)$, $(2, 7, 5)$, $(2, 11, 5)$, $(3, 5, 2)$, $(5, 7, 2)$, $(5, 11, 2)$ of S.

Solution to B3. Yes, f must be differentiable at 0.

We are given that there are real numbers r and s for which

$$\lim_{x\to 0} \frac{f(x)g(x) - f(0)g(0)}{x} = r, \qquad \lim_{x\to 0} \frac{f(x)/g(x) - f(0)/g(0)}{x} = s.$$

Using a common denominator, and by continuity, the latter limit implies

$$\lim_{x\to 0} \frac{f(x)g(0) - f(0)g(x)}{x} = s[g(0)]^2.$$

Adding this to the first limit yields

$$\lim_{x\to 0} (g(x) + g(0)) \frac{f(x) - f(0)}{x} = r + s[g(0)]^2.$$

Because $\lim_{x\to 0}(g(x) + g(0)) = 2g(0) \neq 0$, we have

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \frac{r}{2g(0)} + \frac{sg(0)}{2},$$

and f is differentiable at 0.

Solution to B4. Consider the 2011×2011 matrix S defined by

$$S_{hm} = \begin{cases} 1, & \text{if player } h \text{ won game } m, \\ -i, & \text{if player } h \text{ lost game } m. \end{cases}$$

Let $1 \le h, k \le 2011$. Let n_{hk}^{++}, n_{hk}^{+-}, n_{hk}^{-+}, and n_{hk}^{--}, respectively, denote the number of games won by both h and k, won by h and lost by k, lost by h and won by k, and lost by both h and k. Then

$$\begin{aligned}(T + iW)_{hk} &= n_{hk}^{++} + n_{hk}^{+-}\, i - n_{hk}^{-+}\, i + n_{hk}^{--} \\ &= (S\overline{S}^T)_{hk}.\end{aligned}$$

This shows that $T + iW = S\overline{S}^T$, so that

$$\det(T + iW) = |\det(S)|^2.$$

Let S' be the matrix obtained from S by subtracting the first row multiplied by S_{h1}/S_{11} from the hth row (for $2 \le h \le 2011$), so that $S' = \left(\begin{smallmatrix} S_{11} & * \\ 0 & S'' \end{smallmatrix}\right)$ and $\det(S) = \det(S')$. Note that all entries of S'' are of the form 0, $(\pm 1 \pm i)$, ± 2, or $\pm 2i$, so that $S'' = (1+i)S'''$ for some matrix S''' all of whose entries are 0, ± 1, $\pm i$, or $(\pm 1 \pm i)$. Therefore,

$$\det(S) = \det(S') = S_{11}\det(S'') = (1+i)^{2010} S_{11} \det(S'''), \text{ so}$$

$$|\det(S)|^2 = 2^{2010}|\det(S''')|^2 = 2^{2010}(a^2 + b^2)$$

for some integers a and b, and we are done.

Solution to B5. We are given that

$$\int_{-\infty}^{\infty} \Big(\sum_{i=1}^{n} \frac{1}{1+(x-a_i)^2}\Big)^2 dx = \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \frac{1}{1+(x-a_i)^2} \cdot \frac{1}{1+(x-a_j)^2}\, dx \le An.$$

We first claim that

$$\int_{-\infty}^{\infty} \frac{1}{1+(x-a_i)^2} \cdot \frac{1}{1+(x-a_j)^2}\, dx = \frac{2\pi}{4+(a_i-a_j)^2}.$$

This integral can be evaluated by a more-or-less routine calculation (omitted here) using either partial fractions or contour integration. (Alternatively, it would suffice to establish any lower bound of the form $p/(q + (a_i - a_j)^2)$ with positive p and q for the integral, which can be done by estimating the integrand on a suitable finite interval.)

The first inequality could not hold if more than $n^2/2$ of the summands had integrals larger than $2A/n$. Therefore for at least $n^2/2$ pairs (i, j) we must have

$$2\pi/(4 + (a_i - a_j)^2) \le 2A/n.$$

For those pairs we must have $(4 + (a_i - a_j)^2) \ge n\pi/A$ and hence $(1 + (a_i - a_j)^2) \ge n\pi/(4A)$. Summing over just those pairs, we see that

$$\sum_{i,j=1}^{n} \big(1 + (a_i - a_j)^2\big) \ge \frac{n\pi}{4A} \cdot \frac{n^2}{2} = \frac{\pi}{8A} n^3.$$

Solution to B6. We work over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ with p elements. The problem is equivalent to showing that the $(p-1)$st degree polynomial

$$q(x) = \sum_{k=0}^{p-1} k!\, x^k$$

has at most $(p-1)/2$ distinct roots in $\mathbb{F}_p$. Note that $q(0) = 1$, so 0 is not a root.

Suppose that, on the contrary, $q(x)$ has at least $(p+1)/2$ roots in $\mathbb{F}_p$. Then there exists a monic polynomial of degree $(p-3)/2$ that has among its roots all the nonzero elements of $\mathbb{F}_p$ that are not roots of $q(x)$. That is, there exists

$$f(x) = x^{(p-3)/2} + a_1 x^{(p-5)/2} + \cdots + a_{(p-3)/2}$$

such that all of $1, 2, \ldots, p-1$ are roots of $q(x) f(x)$ in $\mathbb{F}_p$. Because

$$(x-1)(x-2)\cdots(x-(p-1)) = x^{p-1} - 1 \text{ in } \mathbb{F}_p[x],$$

it follows that there is a polynomial $g(x) \in \mathbb{F}_p[x]$ such that

$$q(x) f(x) = (x^{p-1} - 1) g(x).$$

Comparing degrees shows that $g(x)$ has degree $(p-3)/2$. Therefore, the product on the right-hand side has no terms of degrees $(p-1)/2, \ldots, p-2$. Applying this to the product on the left yields

$$j! + (j+1)!\,a_1 + \cdots + (j + (p-3)/2)!\,a_{(p-3)/2} \equiv 0 \pmod{p}$$

for $j = 1, \ldots, (p-1)/2$. We will now get a contradiction by showing that the $(p-1)/2$-by-$(p-1)/2$ matrix

$$\begin{bmatrix} 1! & 2! & \ldots & ((p-1)/2)! \\ 2! & 3! & \ldots & ((p+1)/2)! \\ \vdots & \vdots & \ddots & \vdots \\ ((p-1)/2)! & ((p+1)/2)! & \ldots & (p-2)! \end{bmatrix}$$

is nonsingular over $\mathbb{F}_p$.

In fact, we show more generally, by induction on n, that the $n \times n$ matrix

$$M(n,k) = \begin{bmatrix} k! & (k+1)! & \ldots & (k+n-1)! \\ (k+1)! & (k+2)! & \ldots & (k+n)! \\ \vdots & \vdots & \ddots & \vdots \\ (k+n-1)! & (k+n)! & \ldots & (k+2n-2)! \end{bmatrix},$$

whose i, jth entry is $(k+i+j-2)!$, is nonsingular over $\mathbb{F}_p$ for all $n = 1, 2, \ldots, p$ and $k = 0, \ldots, p-n$. This is clear for $n = 1$. Assume the claim is true for $M(n-1, k)$ with $k = 0, \ldots, p-n+1$. Starting from the matrix $M(n,k)$, we can get zeros in the first column in rows 2 through n by, starting with row n and working up to row 2, subtracting $k+i-1$ times row $i-1$ from row i. The resulting entry in row $i \geq 2$ and column $j \geq 2$ is $(j-1)(k+i+j-3)!$. Dividing out $j-1$ from each column of this $(n-1) \times (n-1)$ submatrix yields the matrix $M(n-1, k+1)$, and it follows that

$$\det M(n,k) = k!\,(n-1)!\,\det M(n-1, k+1) \not\equiv 0 \pmod{p}.$$

Comment. The inductive argument actually shows that in $\mathbb{Z}$,

$$\det M(n,k) = k!\,(k+1)! \cdots (k+n-1)! \cdot (n-1)!\,(n-2)! \cdots 1!.$$

Letters to the Editor

The functions described in William E. Wood's article, "Squigonometry," in the October, 2011 issue of this MAGAZINE, were defined and investigated by me in my paper "A Generalization Of The Trigonometric Functions," which appeared in the December, 1959 issue of the *American Mathematical Monthly*, which I wrote while an undergraduate at the City College of New York [**2**]. In my paper I used a different notation (I called these functions $\alpha_p(x)$ and $\beta_p(x)$). Soon after my paper appeared a reader of the *Monthly*, H. Kaufman, wrote in to say that these functions had previously appeared elsewhere in an 1881 paper and, later, in a 1949 paper by other writers. Kaufman's paper together with these other references appeared in the *Monthly* in October of 1960 [**1**].

In my paper I showed that the functions of even greater index p were periodic and gave some gamma-function values for rational arguments which generalized the Euler formula $\Gamma(-1/2) = \sqrt{\pi}$, the right-hand side being replaced by more complicated radicals involving corresponding constants for my functions. I also showed that for any integer index p, these functions arise from the inversion of hyperelliptic integrals, elliptic for $p = 3$ and 4 (done for $p = 4$ by Cayley), which puts them on a main thoroughfare of mathematics. These functions are used to introduce n-dimensional spherical coordinates in another *Monthly* paper of mine in 1962 [**3**].

DAVID SHELUPSKY
Department of Physics
City College of New York
dshelupsky@gmail.com

REFERENCES

1. H. Kaufman, Remarks on a generalization of the trigonometric functions, *Amer. Math. Monthly* **67** (Oct., 1960) 753. http://dx.doi.org/10.2307/2308650.
2. David Shulupsky, A Generalization Of The Trigonometric Functions, *Amer. Math. Monthly* **66** (Dec., 1959) 879–884. http://dx.doi.org/10.2307/2309789.
3. David Shelupsky, An introduction of spherical coordinates, *Amer. Math. Monthly* **69** (Aug.–Sep., 1962) 644–646. http://dx.doi.org/10.2307/2310836.
4. William E. Wood, Squigonometry, *Math. Mag.* **84** (Oct., 2011) 257–267. http://dx.doi.org/10.4169/math.mag.84.4.257.

Dr. Shelupsky's article is an excellent example of the interesting yet accessible problems that this topic offers—I regret having missed it. Indeed, now that Dr. Shelupsky has provided a starting point for exploring the literature, we find that there have also been some more recent explorations into this topic. For example, among the several articles mentioned in a discussion thread on MathKB.com [**2**] is a 1999 article by Euler and Sadek in this MAGAZINE specifically exploring the generalization of π in the p-norm [**1**].

Functions like these have been studied for quite some time under the heading of elliptic integrals, but it seems the different directions of approach and the many fascinating properties of these "generalized trigonometric functions" have been getting rediscovered every so often in somewhat different contexts since the nineteenth century. I am grateful to Dr. Shelupsky for contributing to and exposing some of this rich history.

WILLIAM E. WOOD
University of Northern Iowa
Cedar Falls, IA 50614
bill.wood@uni.edu

REFERENCES

1. Russell Euler and Jawad Sadek, The πs go full circle, *Math. Mag.* **72** (Feb., 1999) 50–63.
2. Math Knowledge Base, http://www.mathkb.com/Uwe/Forum.aspx/research/4096/Higher-order-trigonometric-functions

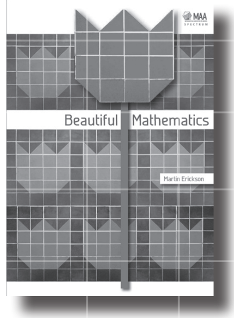
Math. Mag. **85** (2012) 79. doi:10.4169/math.mag.85.1.79.

New from the MAA

Beautiful Mathematics

by Martin Erickson

This book is about beautiful mathematical concepts and creations. Mathematical ideas have an aesthetic appeal that can be appreciated by those who have the time and dedication to investigate. Mathematical topics are presented in the categories of words, images, formulas, theorems, proofs, solutions, and unsolved problems. Readers will investigate exciting mathematical topics ranging from complex numbers to arithmetic progressions, from Alcuin's sequence to the zeta function, and from hypercubes to infinity squared.



Do you know that a lemniscate curve is the circular inversion of a hyperbola? That Sierpinski's triangle has fractal dimension 1.585....? That a regular septagon can be constructed with straightedge, compass, and an angle trisector? Do you know how to prove Lagrange's theorem that every positive integer is the sum of four squares? Can you find the first three digits of the millionth Fibonacci number? Discover the keys to these and many other mathematical problems. In each case, the mathematics is compelling, elegant, simple, and beautiful.

Who should read this book? There is something new for any mathematically-minded person. High school and college students will find motivation for their mathematical studies. Professional mathematicians will find fresh examples of mathematical beauty to pass along to others. Within each chapter, the topics require progressively more prerequisite knowledge. An appendix gives background definitions and theorems, while another gives challenging exercises (with solutions).

Hardback, 2011, 191 pps.	ISBN 978-0-88385-576-8	BTM
List Price: $68.95	MAA Member: $57.95	

To Order Call 1-800-331-1622 or visit us online at www.maa.org

Code: BTMJRNL12

MATHEMATICAL ASSOCIATION OF AMERICA

MAA
MATHEMATICAL ASSOCIATION OF AMERICA 1529 Eighteenth St., NW • Washington, DC 20036

CONTENTS

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS